

On Properly Embedded Minimal Surfaces with Three Ends

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Abstract

We classify all complete embedded minimal surfaces in \mathbb{R}^3 with three ends of genus g and at least $2g+2$ symmetries. The surfaces in this class are the Costa-Hoffman-Meeks surfaces which have $4g+4$ symmetries in the case of a flat middle end. The proof consists of using the symmetry assumptions to deduce the possible Weierstraß data, and then to study the period problems in all cases. To handle the 1-dimensional period problems, we develop a new general method to prove convexity results for period quotients. The 2-dimensional period problems are reduced to the 1-dimensional case by an extremal length argument.

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1 Introduction.

The Costa-Hoffman-Meeks surfaces are the only known properly embedded minimal surfaces in \mathbb{R}^3 with three ends. These are one-parameter families of complete embedded minimal surfaces of genus $k - 1$ for each integer $k > 1$. The following theorem summarizes all that we know about the existence of these surfaces.

Theorem 1 ([6], [10]) *For every $k \geq 2$, there is a one parameter family, $M_{k,x}$, $x \geq 1$, of embedded minimal surfaces of genus $k - 1$ and total curvature $-4\pi(k + 1)$. The surfaces $M_{k,1}$ have two catenoid ends and one flat end. The surfaces $M_{k,x}$, $x \geq 1$, have all three catenoid ends. The Riemann surface $\overline{M}_{k,x}$ is given by:*

$$\overline{M}_{k,x} = \left\{ (z, w) \in \overline{\mathbb{C}}^2 : w^k = -c_x z^{k-1} (z - x) \left(z + \frac{1}{x} \right) \right\}, \quad c_x = (x + x^{-1})^{-1}.$$

The Weierstraß data are:

$$G = \rho \frac{w}{mz + 1}, \quad dh = \frac{z(m + z^{-1})}{(z - x)(z + x^{-1})} dz,$$

for suitable constants ρ and m depending on x (when $x = 1$, $m(1) = 0$).

If $x = 1$, the catenoid ends are located at $(x, 0)$ and $(-x^{-1}, 0)$, and the flat end is (∞, ∞) . If $x > 1$, these three points correspond to catenoid ends.

These surfaces have a rather large symmetry group: In the case of a middle planar end, it consists of the dihedral group Δ_k of order $4k$ generated by a rotation about the x_3 -axis by angle $\frac{\pi}{k}$ followed by a reflection in the (x_1, x_2) -plane, and a reflection in the (x_1, x_3) -plane. In the case of a middle catenoid end, the symmetry group Δ'_k is isomorphic to the dihedral group with $2k$ elements and is generated by a rotation about the x_3 -axis by angle $\frac{2\pi}{k}$, and a reflection in the (x_1, x_3) -plane.

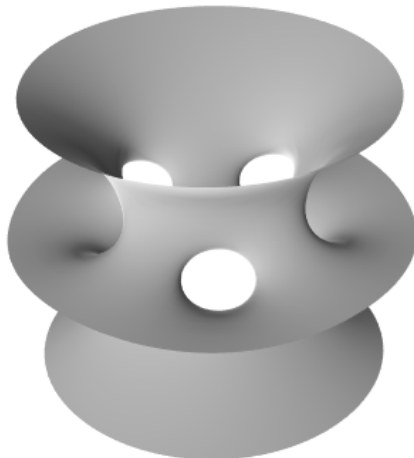


Figure 1: The surface $M_{4,1}$

In [8] Hoffman and Karcher ask whether one can show that these are the only properly embedded, three ended, minimal surfaces with at least this number of symmetries.

In this paper, we give an affirmative answer to this question. More precisely, we will prove

Main Theorem *Let M be a properly embedded minimal surface in \mathbb{R}^3 with three ends, genus $g = k - 1$, $k > 2$, and at least $2k$ symmetries. Then the symmetry group is either Δ_k or Δ'_k , and M is one of the Costa-Hoffman-Meeks surfaces.*

In case $g = 1$, the corresponding result, without any symmetry assumptions, was obtained by Costa in [6].

The proof of the main theorem will follow from the results obtained in Sections 2–4. In fact, the Section 2 is devoted to describing the possible Weierstraß representations of surfaces satisfying the symmetry assumptions. For this, we use similar techniques as developed by Hoffman and Meeks in [9].

As a consequence of this analysis, we obtain three possible Weierstraß representations for such surfaces. To decide the existence of these surfaces, we need to study the period problems in each case.

The first one corresponds to the Costa-Hoffman-Meeks examples, and these examples have been discussed in [8].

The second one corresponds to the so-called Horgan surface (see [15]) with arbitrary dihedral symmetry group. Here, the period problem is 1-dimensional, and the nonexistence of the Horgan surface with 2-fold dihedral symmetry was already proven in [15]. Unfortunately, the method of the simple proof does not extend to the general case, and we use a new technique to prove convexity properties of period quotients. This method relies crucially on the second order differential equation of hypergeometric type satisfied by the period as functions of the moduli. Using this differential equation, we obtain a differential inequality for the second derivative of period quotients which can be integrated to an inequality for the period quotients themselves.

Finally, the third family consists of Weierstraß data describing surfaces of genus 3 with the same symmetry group as the Costa torus with a flat end. Here, the period problem is 2-dimensional. Using an extremal length argument, we are able to reduce this to the second case, concluding also the nonexistence of this family.

The second and third case families consist of Weierstraß data describing rather plausible surfaces whose existence is not obstructed by known results or by the (unproven) Hoffman-Meeks conjecture which states that for embedded minimal surfaces of finite total curvature, the number of ends can be at most the genus plus 2.

For surfaces of the second type one can even produce convincing numerical pictures, because the period problem can be solved to an arbitrary accuracy. For the third case, such a surface would arise from adding two handles to Costa's surface as indicated in figure 2 (the second additional handle is symmetrically placed below the planar end and therefore hidden). It is remarkable that such a handle addition *is* possible in cases with more ends, as shown in [19].

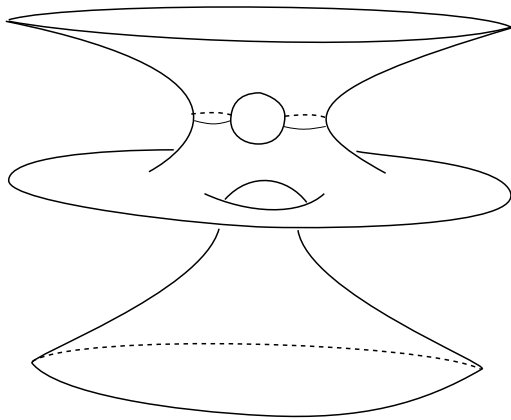


Figure 2: Such a minimal surface does *not* exist

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2 From geometric assumptions to Weierstraß data

The aim of this section is to derive the possible Weierstraß representations for the surfaces which satisfy the hypothesis of the main theorem of this paper.

Let $X : M \rightarrow \mathbb{R}^3$ be a properly minimal embedding of a surface of genus $k - 1$, $k > 2$, with at least three topological ends in three dimensional Euclidean space. As a consequence of Collin's theorem, M has finite total curvature, see [5] or [4] for an alternative proof.

Osserman's Theorem [13] states that M is conformally equivalent to $\overline{M} - \{E_1, E_2, E_3\}$, where \overline{M} is a compact Riemann surface of genus $k - 1$. The removed points correspond to the ends and Osserman proved that the Weierstraß data (G, dh) of X extends meromorphically to \overline{M} . Recall that G is the stereographic projection of the Gauß map.

As $X : M \rightarrow \mathbb{R}^3$ is an embedding then $\mathbb{R}^3 - X(M)$ consists of two connected components. Therefore, up to a rigid motion, G must alternate between 0 and ∞ on the ends of M , ordered by height in \mathbb{R}^3 . So, we can assume without loss of generality that $G(E_1) = G(E_3) = 0$, $G(E_2) = \infty$, and that E_1 is the highest end, E_2 is the middle end and E_3 is the lowest end

By the maximum principle, the top and bottom ends must be *catenoid* ends.

We denote by $Iso(M)$ the isometry group of M and by $Sym(M)$ the subgroup of $Iso(M)$ consisting of isometries which are the restriction of a rigid motion in \mathbb{R}^3 leaving $X(M)$ invariant. Choi, Meeks and White [3] have proved that a properly embedded minimal surface M in \mathbb{R}^3 which has more than one end is minimally rigid. In particular any intrinsic isometry of M extends to an isometry of \mathbb{R}^3 .

We assume that $Sym(M) = Iso(M)$ has at least $2k$ elements. A symmetry of $X(M)$ induces in a natural way a conformal automorphism of M which extends to \overline{M} leaving the set $\{E_1, E_2, E_3\}$ invariant. Using Hurwitz's Theorem, $Sym(M)$ is finite. Thus, after fixing a suitable origin, $Sym(M)$ is a finite group Δ of orthogonal linear transformations of \mathbb{R}^3 . Furthermore, since the normal vectors at the ends are vertical, Δ leaves the x_3 -axis invariant.

In what follows we will not distinguish between a symmetry $F \in \Delta$ of M and the induced conformal automorphism $F|_{X(M)}$.

Observe that any symmetry in Δ leaves the set $\{E_1, E_3\}$ invariant and fixes the point E_2 . Let Δ_0 be the subgroup of holomorphic transformations in Δ , and denote by $\mathcal{R} \subseteq \Delta_0$ the cyclic subgroup of rotations around the x_3 -axis. It is obvious that

$$[\Delta : \Delta_0] \leq 2, \quad [\Delta_0 : \mathcal{R}] \leq 2. \tag{1}$$

Denote by R the rotation around the x_3 -axis with the smallest positive angle in Δ_0 . Then $\mathcal{R} = \langle R \rangle$.

In this setting, López, Rodríguez and the first author have proved the

Theorem 2 ([11]) *If $\text{ord}(R) \geq k$ then there exists $x \geq 1$ such that, up to conformal transformations and rigid motions in \mathbb{R}^3 , $M = M_{k,x}$ and $X = X_{m(x)}$ is the minimal embedding described in Theorem 1.*

Let D be conformal disk in \overline{M} centered at E_2 which is invariant under Δ_0 . Then $\{F|_D : F \in \Delta_0\}$ is a finite group of biholomorphisms of the disk fixing the origin. Hence this group is cyclic, generated by some automorphism J . As a symmetry, J is either a rotation around the x_3 -axis or a rotation around the x_3 -axis followed by a symmetry with respect to the (x_1, x_2) -plane.

Hence in the rest of this section we can assume that J is the generator of Δ_0 corresponding either to a rotation around the x_3 -axis by an angle of $\frac{2\pi}{\text{ord}(J)}$, or to a rotation around the x_3 -axis by an angle of $\frac{2\pi}{\text{ord}(J)}$ followed by a symmetry with respect to the (x_1, x_2) -plane, where $\text{ord}(J)$ is the order of J and $\text{ord}(J) = |\Delta_0|$. Note that either $R = J$ or $R = J^2$.

We introduce the following notation: for any point $Q \in \overline{M}$ denote by

$$I(Q) = \{F \in \Delta_0 : F(Q) = Q\}$$

the isotropy group of Q in Δ_0 and denote by $\mu(Q) = |I(Q)|$ the cardinality of $I(Q)$. We also denote by $\text{orb}(Q) = \{Q, J(Q), \dots, J^{|\Delta_0|-1}(Q)\}$ the orbit of Q under Δ_0 . Notice that $\text{orb}(Q)$ has $\frac{|\Delta_0|}{\mu(Q)}$ elements.

If E_1, E_2 and E_3 are all catenoid ends, then $\text{Sym}(M)$ does not contain any rotation followed by a symmetry, i. e. $R = J$. Hence, from Theorem 2 we get the following

Corollary 1 ([11]) *If $X : M \rightarrow \mathbb{R}^3$ has three catenoid ends and $\text{Sym}(M)$ contains $2k$ elements or more, then up to natural transformations $X = X_{m(x)}$, $x > 1$.*

Therefore we only need to deal with the following case:

$$R = J^2, E_2 \text{ is a flat end, and } E_1 \text{ and } E_3 \text{ are catenoid ends.}$$

Thus Δ_0 is generated by a rotation followed by a symmetry. From the Riemann-Hurwitz formula we obtain

$$4 - 2k = |\Delta_0| \chi(\overline{M}/\Delta_0) - (2|\Delta_0| - 3 + \sum_{Q \in M} (\mu(Q) - 1))$$

Since $|\Delta_0| \geq k$ we deduce $\chi(\overline{M}/\Delta_0) > 0$ and so \overline{M}/Δ_0 is a sphere and in fact $\chi(\overline{M}/\Delta_0) = 2$. Using this in the above formula, we get

$$\sum_{Q \in M} (\mu(Q) - 1) = 2k - 1 \tag{2}$$

Let $orb(Q_1), \dots, orb(Q_s)$ be the different nontrivial orbits of Δ_0 on M (i.e., $\mu(Q_i) > 1$, $i = 1, \dots, s$ and if $Q \in M - \cup_{i=1}^s orb(Q_i)$ then $\mu(Q) = 1$). If we let $m_i = \frac{|\Delta_0|}{\mu(Q_i)}$, $i = 1, \dots, s$, then (2) gives

$$\sum_{i=1}^s (|\Delta_0| - m_i) = 2k - 1 \quad (3)$$

Since $|\Delta_0|$ is even then at least one of the numbers m_i is odd. On the other hand, if m_i is odd then J^{m_i} is a rotation around the x_3 -axis followed by a symmetry with respect to the (x_1, x_2) -plane which only fixes the origin of \mathbb{R}^3 . As X is an embedding, there is at most one point of M mapped by X into the origin, and so only one m_i is odd and $m_i = 1$. Up to re-indexing we can assume that $m_1 = 1$.

Therefore m_i is even, $i \geq 2$, and J^{m_i} is a rotation, $i \geq 2$. If $m_i > 2$, $i \in \{2, \dots, s\}$, then Q_i and $J^2(Q_i)$ are two different points lying in $orb(Q_i)$. Since J^{m_i} is a rotation around the x_3 -axis then $X(Q_i)$ and $X(J^2(Q_i))$ lie in the x_3 -axis. Moreover J^2 is a rotation around the x_3 -axis too, and thus $X(Q_i) = X(J^2(Q_i))$, which contradicts the fact that X is an embedding. Hence $m_i = 2 \forall i \geq 2$ and (3) becomes

$$|\Delta_0| + (s - 1)(|\Delta_0| - 2) = 2k$$

If $s = 1$ we get $|\Delta_0| = 2k$, and by Theorem 2 the surface corresponds to the unique element of the Costa-Hoffman-Meeks family with a flat end.

If $s > 1$, using the Riemann-Hurwitz formula once again, we get

$$|\mathcal{R}| \chi(\overline{M}/\mathcal{R}) = \chi(\overline{M}) + (3 + 2s - 1)(|\mathcal{R}| - 1) = 4 - 2k + 2(s + 1)(|\mathcal{R}| - 1).$$

As $s > 1$, we obtain $\chi(\overline{M}/\mathcal{R}) = 2$. So, the above equality gives:

$$\frac{k}{2} \leq |\mathcal{R}| = 1 + \frac{k - 1}{s} \leq \frac{k + 1}{2}.$$

This leaves us with two possibilities:

- (A) $|\mathcal{R}| = (k + 1)/2$. In this case $k - 1$ is even and $s = 2$. Then, we have three points in M which are mapped by X into the x_3 -axis: Q_1 , Q_2 and $Q_3 = J(Q_2)$.
- (B) $|\mathcal{R}| = k/2$, which implies $k - 1 = 3$ and $s = 3$. Now, there are five points in M which are mapped by X into the x_3 -axis: Q_1 , Q_2 , Q_3 , $Q_4 = J(Q_2)$ and $Q_5 = J(Q_3)$.

Observe that in both cases any symmetry of \overline{M} fixes the point Q_1 . Moreover, any symmetry of \overline{M} which fixes the ends must fix the points Q_i , $i \geq 2$, and any symmetry of \overline{M} which interchanges the catenoid ends must interchange the points Q_i and $J(Q_i)$, $i \geq 2$.

At this point we focus on the branched covering $z : \overline{M} \rightarrow \overline{M}/\mathcal{R} \equiv \overline{\mathcal{C}}$. Up to Möbius transformations we can assume in both cases that: $z(E_2) = \infty$, $z(Q_1) = 0$ and $z(Q_2) = 1$.

From (1) and the hypothesis $|\Delta| \geq 2k$ we see that $[\Delta : \Delta_0] = 2$, $[\Delta_0 : \mathcal{R}] = 2$. So Δ/\mathcal{R} is a group of conformal transformations of $\overline{\mathbb{C}}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence it follows that $\Delta/\mathcal{R} = \{R, f, g, h\}$, where:

1. f is a holomorphic involution fixing 0 and ∞ , so $f(z) = -z$ and $z(J(Q_2)) = -1$.
2. g is an antiholomorphic involution fixing the real axis, and so $g(z) = \bar{z}$.
3. $h(z) = (f \circ g)(z) = -\bar{z}$.

Taking these facts into account, we deduce that:

(A) $\lambda = z(E_1) \in \mathbb{R} - \{0, 1\}$ and $z(E_3) = -\lambda$.

(B) $\lambda = z(E_1) \in \mathbb{R} - \{0, 1\}$, $z(E_3) = -\lambda$, $\mu = z(Q_3) \in \mathbb{R} - \{0, 1, \lambda\}$ and $z(Q_5) = -\mu$.

We now investigate the behavior of the Gauß map at an end and at a point on the x_3 -axis in the presence of the rotational symmetry R . Each catenoid end still produces a simple zero of G . However, the flat end must correspond to a pole of order $jm + 1$, where j is a positive integer and m is the order of R . On the other hand, if Q is a point of the x_3 -axis, then the order of Q as zero or pole of G is $jm - 1$, for a positive integer j (for details see [2]).

Using these arguments and the fact that $\deg(G) = k + 1$, we are able to deduce that the divisor associated to G as a meromorphic function on \overline{M} is (in each case):

(A)
$$(G) = \frac{E_1 \cdot E_3 \cdot Q_2^{\frac{k-1}{2}} \cdot Q_3^{\frac{k-1}{2}}}{E_2^{\frac{k+3}{2}} \cdot Q_1^{\frac{k-1}{2}}}.$$

(B)
$$(G) = \frac{E_1 \cdot E_3 \cdot Q_1 \cdot Q_3 \cdot Q_5}{E_2^3 \cdot Q_2 \cdot Q_4}.$$

Next we write down the divisor for dh on \overline{M} . The behavior of dh is determined by the rule that on M , $1/G dh$ is holomorphic with zeroes precisely at the poles of G , but with double order. Recalling that the poles and zeroes of $1/G dh$ can lie only at the poles and zeroes of G and their order is determined by the requirement that the first and second Weierstraß forms have poles of order two, we obtain:

(A)
$$(dh) = \frac{Q_1^{\frac{k-1}{2}} \cdot Q_2^{\frac{k-1}{2}} \cdot Q_3^{\frac{k-1}{2}} \cdot E_2^{\frac{k-1}{2}}}{E_1 \cdot E_3}.$$

(B)
$$(dh) = \frac{E_2 \cdot \prod_{i=1}^5 Q_i}{E_1 \cdot E_3}.$$

We will now use similar techniques to those introduced by Hoffman and Meeks in [9] to obtain the two possible conformal structures for \overline{M} .

Case (A). Define $N = M - \{Q_1, Q_2, Q_3\}$ and denote by $n = (k + 1)/2$. Then $z_{|N} : N \rightarrow \mathbb{C} - \{0, \pm 1, \pm \lambda\}$ is a n -fold unbranched cyclic covering. Moreover, the conformal structure on N determines that \overline{M} . We may determine $z_{|N}$ as follows. Recall that R is the generator of \mathcal{R} corresponding to counterclockwise rotation around the x_3 -axis by an angle of $\frac{2\pi}{n}$. Let α_i , $i = 1, \dots, 5$, be a counterclockwise circuit around 0 , -1 , 1 , $-\lambda$ and λ respectively, and $\widehat{\alpha}_i$ its lift to N . The endpoints of $\widehat{\alpha}_i$ will differ by a deck transformation of the form R^{k_i} , $0 \leq k_i \leq n - 1$, $i = 1, \dots, 5$. The choice of R and the fact that we have oriented \overline{M} with downward-pointing normals at E_1 , E_3 , Q_2 and Q_3 implies that R has rotation number $\frac{\pi}{n}$ at E_1 and E_3 and rotation number $-\frac{\pi}{n}$ at Q_2 and Q_3 . Using similar arguments for Q_1 , R has rotation number $\frac{\pi}{n}$ at Q_1 . Hence $k_1 \equiv k_4 \equiv k_5 \equiv 1 \pmod{n}$ and $k_2 \equiv k_3 \equiv -1 \pmod{n}$. The numbers k_1, \dots, k_5 determine the induced map from the fundamental group $\Pi_1(\mathbb{C} - \{0, \pm 1, \pm \lambda\})$ onto \mathbb{Z}_n whose kernel corresponds to $z_*(\Pi_1(N)) \subset \Pi_1(\mathbb{C} - \{0, \pm 1, \pm \lambda\})$. Any n -fold cyclic covering of $\mathbb{C} - \{0, \pm 1, \pm \lambda\}$ is equivalent to $z_{|N}$ if the associated representation has the same kernel. In particular the cyclic covering defined by the z -projection of

$$\left\{ (z, w) \in (\mathbb{C} - \{0, \pm 1, \pm \lambda\}) \times (\mathbb{C} - \{0\}) : w^n = \frac{z(z^2 - \lambda^2)}{z^2 - 1} \right\}$$

is equivalent to $z_{|N}$. The extension of this covering to the Riemann surface

$$\overline{M}_{n\lambda} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 : w^n = \frac{z(z^2 - \lambda^2)}{z^2 - 1} \right\}$$

is conformally equivalent to $z : \overline{M} \rightarrow \overline{M}/\mathcal{R}$. In particular $\overline{M} = \overline{M}_{n\lambda}$, $E_1 = (\lambda, 0)$, $E_2 = (\infty, \infty)$, $E_3 = (-\lambda, 0)$, $Q_1 = (0, 0)$, $Q_2 = (1, \infty)$ and $Q_3 = (-1, \infty)$. Furthermore, the conformal transformation R corresponds to $R(z, w) = (z, e^{\frac{2\pi i}{n}} w)$. We denote by $M_{n\lambda} = \overline{M}_{n\lambda} - \{E_1, E_2, E_3\}$.

Taking into account the expression for (G) and (dh) that we have obtained before, we deduce that:

$$G = \rho \frac{z^2 - \lambda^2}{w^{n-1}}, \quad dh = \sigma \frac{dz}{z^2 - \lambda^2},$$

where, up to scaling and rigid motion, $\rho \in \mathbb{R}^*$ and $\sigma \in \mathbb{C}^*$, $|\sigma| = 1$. The 1-form dh is the pull-back of a differential on $\overline{\mathbb{C}}$. So, the period problem reduces to impose that its residues at the ends are real. This leads to $\sigma = \pm 1$. Up to a symmetry, we can assume $\sigma = 1$.

For λ , we have to consider the two cases $\lambda \in (0, 1)$ and $\lambda \in (1, \infty)$. The former can be easily excluded as follows (see also section 3).

Assume the contrary, that is, $\lambda < 1$. Denote by $S : \overline{M}_{n,\lambda} \rightarrow \overline{M}_{n,\lambda}$ the anti-conformal map $S(z, w) = (\bar{z}, \bar{w})$. It is obvious that $S^*(\Phi_1, \Phi_2, \Phi_3) = (\overline{\Phi_1}, -\overline{\Phi_2}, \overline{\Phi_3})$, which means that S induces in $X(M)$ a reflection in the plane $x_2 = 0$.

Let γ_1, γ_2 and γ_3 be the lifts to $\overline{M}_{n,\lambda}$ of the segments in the z -plane: $[1, \infty[$, $[0, \lambda[$ and $[-1, -\lambda[$, respectively, and label $\Gamma_i = X(\gamma_i)$, $i = 1, 2, 3$. Using the above reasoning Γ_i is contained in $x_2 = 0$, $i = 1, 2, 3$.

On the other hand, one has that:

- Γ_1 is a curve connecting the $X(Q_2)$ (in the x_3 -axis) with the flat end.
- Γ_2 is another divergent curve connecting $X(Q_1) = (0, 0, 0)$ with the catenoid end E_1 .
- Γ_3 connects the point $X(Q_3)$ and the catenoid end E_3 .

These facts lead to $\sharp(\cap_{i=1}^3 \Gamma_i) \geq 2$ (see Figure 3), which is contrary to the embeddedness. So, we conclude that λ is in $]1, +\infty[$.

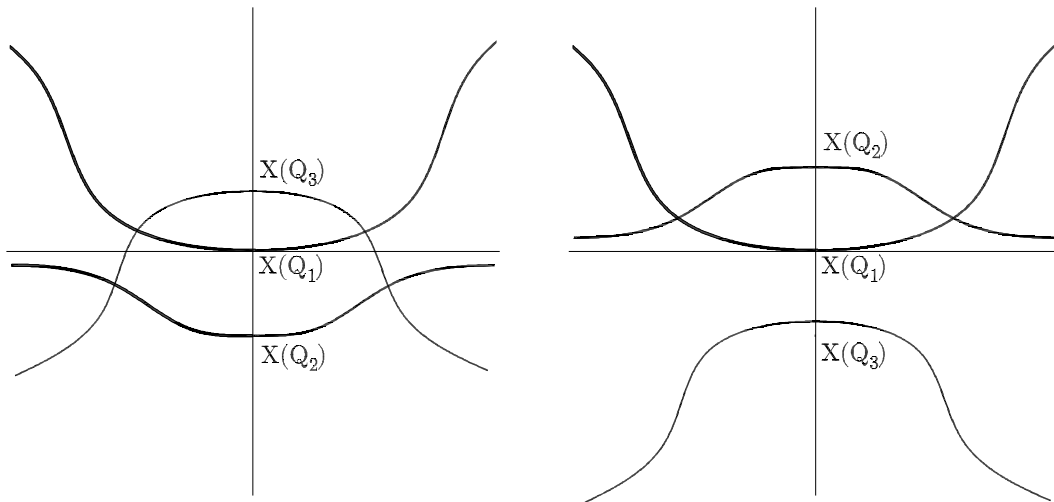


Figure 3: The two possible situations for the curves Γ_i , $i = 1, 2, 3$.

Case (B). Recall that, in this case, the genus of \overline{M} is 3 and $ord(R) = 2$. Using similar arguments as in case (A), we deduce that \overline{M} is conformally equivalent to

$$\overline{M}_{\lambda,\mu} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 : w^2 = z(z^2 - 1)(z^2 - \lambda^2)(z^2 - \mu^2) \right\},$$

and the Weierstraß data are:

$$G = \rho \frac{w}{z^2 - 1}, \quad dh = \sigma \frac{dz}{z^2 - \lambda^2},$$

where $\rho \in \mathbb{R}^*$ and $\sigma \in \mathbb{C}^*$, $|\sigma| = 1$. The period conditions for dh say us that σ is real and, once again, we can suppose that $\sigma = 1$.

3 Preliminary discussion of the period problem

In this section, we will discuss the period condition on the candidate Weierstraß data from the last section. The period condition ensures that the potentially multiple valued surface parametrization is in fact single valued. It can be stated as follows:

For any closed cycle γ on X , we need

$$\operatorname{Re} \int_{\gamma} dh = 0 \quad (4)$$

$$\int_{\gamma} Gdh = \overline{\int_{\gamma} \frac{1}{G} dh} \quad (5)$$

Our goal will be to show that in both cases **A** and **B**, the period problem can not be solved.

As we mentioned at the end of Section 2, the period condition on dh is equivalent to the requirement that σ is real. We will without loss of generality assume that $\sigma = 1$.

The discussion of the second condition is much more difficult. In the first case **A**, it requires a detailed analysis of hypergeometric functions, which is postponed to section 4. In case **B**, the period problem appears to be more difficult than in case **A**, as it depends on one additional parameter. Surprisingly, it can in fact be reduced to case **A** by a simple extremal length argument, which is done in section 5. For both discussions, it is helpful to consider certain flat polygonal domains associated naturally to the Weierstraß forms Gdh and $\frac{1}{G}dh$. These domains have been introduced in [18, 19] and have also been used in [17].

We illustrate the concept in case **A** where the Weierstraß forms, restricted to the upper half plane in the z -sphere, are given by

$$\begin{aligned} Gdh &= \rho \frac{dz}{w^{n-1}} = \rho z^{\frac{1}{n}-1} (z^2 - 1)^{1-\frac{1}{n}} (z^2 - \lambda^2)^{\frac{1}{n}-1} dz \\ \frac{1}{G}dh &= \frac{1}{\rho} \frac{w^{n-1} dz}{(z^2 - \lambda^2)^2} = \frac{1}{\rho} z^{1-\frac{1}{n}} (z^2 - 1)^{\frac{1}{n}-1} (z^2 - \lambda^2)^{-1-\frac{1}{n}} dz \end{aligned}$$

We now look at the maps $\mathbb{H} \rightarrow \mathbb{C}$ defined by

$$z \mapsto \int^z Gdh \quad \text{and} \quad z \mapsto \int^z \frac{1}{G}dh.$$

From the Schwarz-Christoffel formula (see [12]) we see that these holomorphic maps map the upper half plane conformally onto a (possibly branched and unbounded) euclidean polygon bounding a disk. The zeroes and poles of the integrands, that is, $0, \pm 1, \pm \lambda$, and ∞ are mapped to the vertices of these domains, and the angles are determined explicitly by the exponents of the factors.

The following figure shows the two image domains in case **A** with $n = 2$ and $\lambda > 1$. Here and in the following figures, the left domain will correspond to Gdh , and the right domain to $\frac{1}{G}dh$.

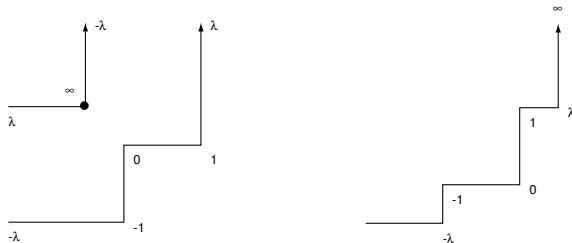


Figure 4: Case Aa

The polygonal arc is the image of the real axes, the arrow(s) indicate the orientation, and the domains are always to the left of these polygonal boundaries. Interior branched points are indicated by a fat dot. Such branched domains can be realized by taking a domain for each boundary component, slitting these domains with slits emanating from the branched points, and glueing them together.

As the minimal surface patch corresponding to the upper half plane is the conformal image of the upper half plane, the above domains are other conformally correct representations of this surface patch.

Moreover, some of the surface symmetries are visible: surface isometries which preserve the absolute value of Gdh become automorphisms of the new flat surface metric $|Gdh|$, and their fixed point sets will become straight lines. In the particular case of this example, the reflections about the vertical coordinate planes decompose the minimal surface into four simply connected isometric pieces, each the image of the upper half plane. The symmetry lines become just the polygonal segments. There is an additional assumed symmetry, namely a rotation about a horizontal line on the surface, also preserving $|Gdh|$. This symmetry becomes the reflectional of the domains. Our convention here is that the figures are symmetric with respect to the $y = -x$ diagonal. This can always be achieved by a rotation. So the $y = -x$ axes takes over the role of the imaginary axes, and hence conjugation will mean reflection at the $y = x$ axes.

The periods of the meromorphic 1-forms Gdh and $\frac{1}{G}dh$ can now be read off from these domains as displacement vectors between parallel edges, because integrating (say) Gdh over a closed cycle is nothing but developing the curve using the flat metric. The period condition on Gdh and $\frac{1}{G}dh$ requires that their respective periods are complex conjugate.

Under the chosen normalization of the Schwarz-Christoffel maps, this means that corresponding period vectors (that is, displacement vectors between the same pair of parallel edges) in Gdh and $\frac{1}{G}dh$ must be conjugate by a reflection at the $y = x$ diagonal.

Definition 1 *Two period domains as above are called conjugate if all corresponding period vectors are symmetric with respect to reflection at the $y = x$ axes.*

Given a conjugate pair of domains, it is not clear a priori that they come from the same modulus, i.e. that the modulus λ is the same number in both Schwarz-Christoffel maps. If this is the case, we call the two domains conformal:

Definition 2 *Two period domains as above are called conformal if there is a conformal biholomorphism mapping vertices to corresponding vertices.*

Using terminology, we can rephrase the period problem as follows:

Theorem 3 *The period problem in case **A** or **B** has a solution if and only if there is a pair of conjugate and conformal domains which corresponds to the respective Weierstraß data.*

Using this observation, we can give an alternative argument to show that in case **A** $\lambda \in (0, 1)$ is impossible. Here the domains look as in figure 5.

Notice how the angle geometry of this domain imposes restrictions on the periods: The period vector corresponding to the cycle connecting $(0, \lambda)$ to $(1, \infty)$ has to point up in the Gdh domain, because the branched point cannot lie outside the domain, and the period vector points left in the $\frac{1}{G}dh$ domain by the angles. This contradicts the conjugacy requirement for the period vectors (recalling that the real axes in our figures is the $y = x$ diagonal).

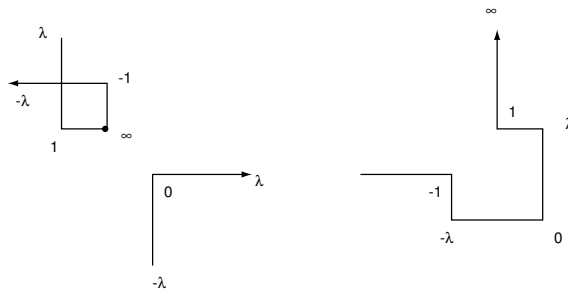


Figure 5: Case Ab

In the case of higher dihedral symmetry, we obtain the kind of domains as in figure 6.

The periods can be read off as before. Notice that the period vectors are always parallel to one type of edges. Again the imaginary axes becomes the symmetry diagonal.

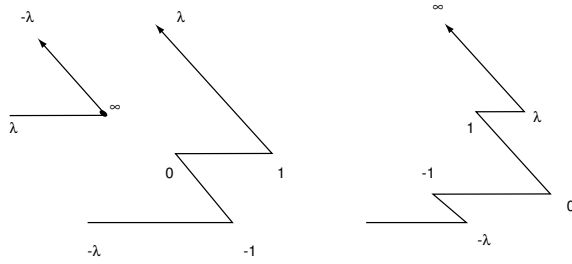


Figure 6: Case Aa, $n > 2$

For case B, we have to distinguish six different cases corresponding to the relative position of $0, 1, \lambda, \mu$. Fortunately, all but two drop out by the same type of argument as before (see the figures 7–12).

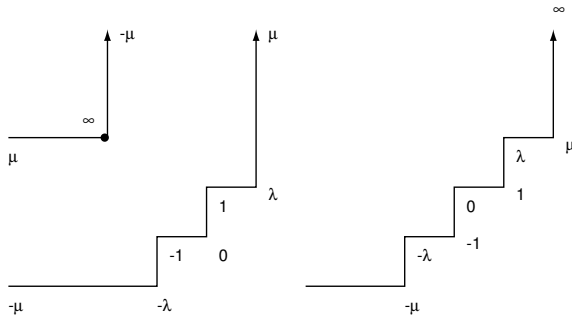


Figure 7: Case Ba: $0 < 1 < \lambda < \mu$

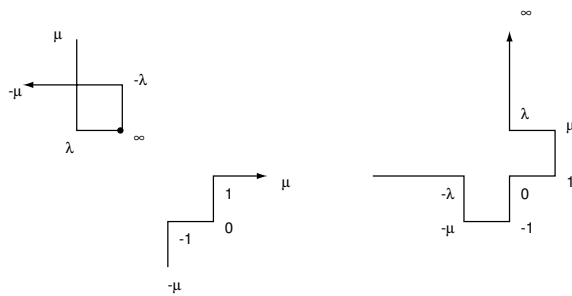


Figure 8: Case Bb: $0 < 1 < \mu < \lambda$

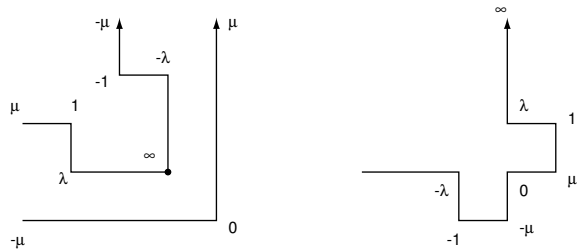


Figure 9: Case Bc: $0 < \mu < 1 < \lambda$

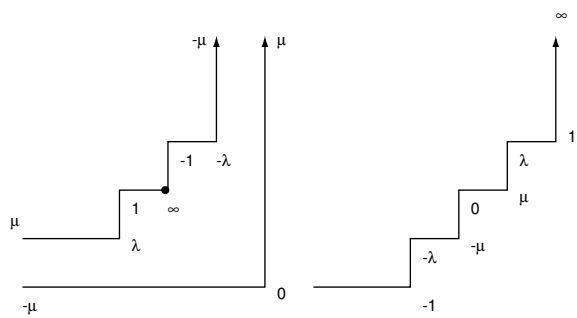


Figure 10: Case Bd: $0 < \mu < \lambda < 1$

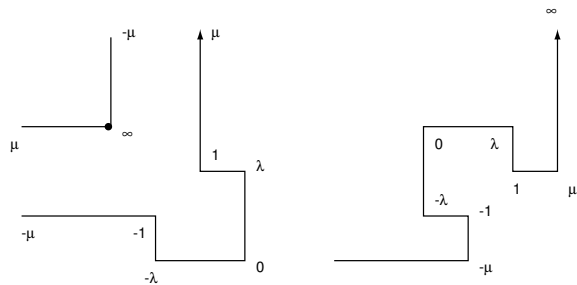


Figure 11: Case Be: $0 < \lambda < 1 < \mu$

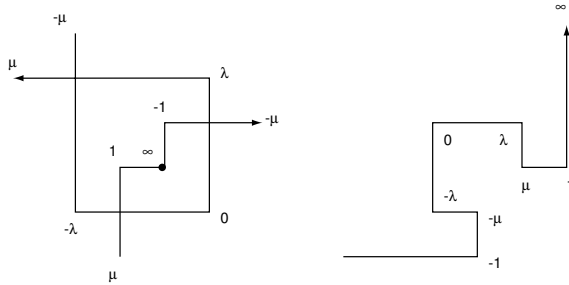


Figure 12: Case Bf: $0 < \lambda < \mu < 1$

Cases b,c,d,f are impossible, because the period vectors have inconsistent signs: To ensure conjugacy of the domains, the lowest edge of the $\frac{1}{G}$ domain needs to be the leftmost edge of the Gdh domain. Similarly, the ordering of the horizontal edges from bottom to top $\frac{1}{G}$ imposed by the geometry of the domain has to persist as an ordering from left to right in the Gdh domain. Using this together with the assumed reflectional symmetry of the domains, this leads in all these cases to domains which are geometrically impossible.

Cases a and e will be dealt with in section 5.

This concludes our preliminary discussion of the period problem.

4 Convexity properties of period quotients

In this section, we will study period quotients of meromorphic 1-forms as functions of one real modulus $\lambda \in (1, \infty)$. Such period quotients arise naturally in existence and classification problems for minimal surfaces (see [16]). The reason for this is that the second period condition in 5 asks for complex conjugate period vectors, and often this can be decided by just asking for proportional period vectors. In situations with only one free essential parameter, this reduces to period quotient functions.

However usually it is a delicate problem to find good estimates for the transcendental functions arising this way. The difficulties are caused by two different reasons: First, the periods forming the fraction are integrals over different cycles on the Riemann surface, giving no idea how to compare the integrals. Secondly, despite the fact that the integrands are real 1-forms, very often the cycles are curves in some complex domain which cannot be replaced by better estimable integrals along segments on the real axes.

Both these problems are overcome here by proving a much stronger statement than needed: The ultimate goal here will be to show that two such period quotients can only be equal for $\lambda = 1$. We will do this by proving that one of these functions is convex, the other concave and their graphs touch at $\lambda = 1$.

The reason why it is actually simpler to control second derivatives than the functions

themselves lies in the fact that period integrals often satisfy linear differential equations. In our case we obtain a formula relating the second derivative of a period quotient to the coefficients of the differential equation and a term involving only one period and its derivative. The latter term can easily be estimated using direct computations.

All the discussions in this section are directly concerned with case **A**. Hence we consider the meromorphic 1-forms from case **(A)**:

$$\begin{aligned}\omega_1 &= z^{\frac{1}{n}-1}(z^2-1)^{1-\frac{1}{n}}(\lambda^2-z^2)^{\frac{1}{n}-1}dz \\ \omega_2 &= z^{1-\frac{1}{n}}(z^2-1)^{-1+\frac{1}{n}}(\lambda^2-z^2)^{-\frac{1}{n}-1}dz\end{aligned}$$

which are single-valued on the Riemann surface X defined by the algebraic equation

$$w^n = \frac{z(z^2 - \lambda^2)}{z^2 - 1}$$

As explained in section 2, we can assume that $\lambda > 1$.

The principal goal is to prove

Theorem 4 *There is no complex number $\rho \in \mathbb{C}^*$ such that we have for all closed cycles γ on X*

$$\int_{\gamma} \rho \omega_1 = \overline{\int_{\gamma} \frac{1}{\rho} \omega_2}$$

To prove this, we will first reduce this statement to a period quotient statement for real integrals. Observe that ω_1 can be integrated safely on the intervals $[0, 1]$ and $[1, \lambda]$. This does not hold for ω_2 , and we take care of this by

Lemma 1

$$\int_{\gamma} \omega_2 = -\frac{1}{2(\lambda^2-1)\lambda^{3/n}} \int_{\gamma^*} \omega_3$$

where

$$\omega_3 = z^{\frac{1}{n}-1}(z^2-1)^{-\frac{1}{n}}(\lambda^2-z^2)^{\frac{1}{n}}dz$$

and γ^* denotes the cycle obtained from γ by applying the conformal transformation $S(z, w) = (\lambda/z, \lambda^{\frac{3}{n}}/w)$.

Proof: Using first partial integration, we obtain

$$\int_{\gamma} \omega_2 = \int_{\gamma} \left[\omega_2 + \frac{n}{2(\lambda^2-1)} d\left(\frac{1}{w}\right) \right] = \frac{1}{2(\lambda^2-1)} \int_{\gamma} \frac{dz}{zw}. \quad (6)$$

So, we have:

$$\int_{\gamma} \omega_2 = \int_{S_*\gamma} S^* \left(\frac{dz}{zw} \right) = \frac{1}{2(\lambda^2-1)\lambda^{3/n}} \int_{\gamma^*} \omega_3.$$

□

So, using (6) and (7), one has:

$$\begin{aligned}
\int_{\alpha} \omega_2 &= \int_{T_{*,\alpha}} T^* \left(\frac{dz}{z w} \right) = \frac{1}{2(\lambda^2 - 1)\lambda^{3/n}} \int_{\alpha+\beta} \overline{\omega_3} = \\
&= \frac{1}{2(\lambda^2 - 1)\lambda^{3/n}} (e^{-\frac{2\pi i}{n}} - 1) \left[\int_{-\lambda}^{-1} \frac{|w|}{z} dz + \int_0^1 \frac{|w|}{z} dz \right] = \\
&= \frac{1}{2(\lambda^2 - 1)\lambda^{3/n}} (e^{-\frac{2\pi i}{n}} - 1) \left[\int_0^1 \frac{|w|}{z} dz - \int_1^{\lambda} \frac{|w|}{z} dz \right].
\end{aligned}$$

□

Now introduce the following period integrals:

$$\begin{aligned}
f_1(\lambda) &= \int_0^1 z^{\frac{1}{n}-1} (1-z^2)^{1-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n}-1} dz \\
f_2(\lambda) &= \int_1^{\lambda} z^{\frac{1}{n}-1} (z^2-1)^{1-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n}-1} dz \\
g_1(\lambda) &= \int_0^1 z^{\frac{1}{n}-1} (1-z^2)^{-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n}} dz \\
g_2(\lambda) &= \int_1^{\lambda} z^{\frac{1}{n}-1} (z^2-1)^{-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n}} dz
\end{aligned}$$

and the associated period quotients

$$\text{pq}_f(\lambda) = f_2(\lambda)/f_1(\lambda) \quad \text{pq}_g(\lambda) = g_2(\lambda)/(g_1(\lambda) - g_2(\lambda))$$

Later we will need

Lemma 3

$$\begin{aligned}
g_1(\lambda) &> g_2(\lambda) \\
g'_1(\lambda) &> g'_2(\lambda)
\end{aligned}$$

Proof: The first inequality is a simple consequence of lemma 2. In fact, the substitution $v = \lambda/z$ carries the cycle from 0 to 1 into a cycle homotopic to the difference of the cycles from 0 to 1 and 1 to λ , so that this difference can be evaluated as an integral over a positive integrand. The only critical issue here is that the integrals need to converge. We can apply a similar strategy for the second inequality. Here we compute

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \int_{\gamma^*} \omega_3 &= \frac{2\lambda}{n} \int_{\gamma^*} v^{\frac{1}{n}-1} (1-v^2)^{-\frac{1}{n}} (\lambda^2 - v^2)^{\frac{1}{n}} dv \\
&= -\frac{2\lambda}{n} \int_{\gamma} \left(\frac{\lambda}{z}\right)^{\frac{1}{n}-1} \left(1 - \frac{\lambda^2}{z^2}\right)^{-\frac{1}{n}} \left(\lambda^2 - \frac{\lambda^2}{z^2}\right)^{\frac{1}{n}-1} \frac{\lambda}{z^2} dz \\
&= -\frac{2\lambda^{3-\frac{3}{n}}}{n} \int_{\gamma} z^{1-\frac{1}{n}} (z^2 - 1)^{\frac{1}{n}-1} (z^2 - \lambda^2)^{-\frac{1}{n}} dz
\end{aligned}$$

The last integral converges, and the same reasoning about the cycles as above implies that $g'_1(\lambda) > g'_2(\lambda)$. \square

We now come to the main theorem of this section. The following lemma 4 shows that this is a sharp estimate. With other words, for λ close to 1 one gets a minimal surface with its periods *almost* closed.

Theorem 5 *For every $\lambda > 1$, we have $\text{pq}_f(\lambda) > \text{pq}_g(\lambda)$.*

Assuming theorem 5, the theorem 4 follows easily. The theorem 5 is proven as a corollary of lemma 4 and proposition 1:

Lemma 4 *The period quotient functions satisfy*

$$\begin{aligned}
\text{pq}_f(1) &= \text{pq}_g(1) = 0 \\
\text{pq}'_f(1) &= \frac{n-1}{n^2 \sin \frac{\pi}{n}} \pi \\
\text{pq}'_g(1) &= \frac{1}{n^2 \sin \frac{\pi}{n}} \pi
\end{aligned}$$

Proof: The first statement follows instantly from the definitions. For the others, we first deal with $\text{pq}'_f(1)$. For this, we need to evaluate $f'_2(1)$, and to compute the derivative, we have to ensure that the integral converges. This is done using the substitution $x = \lambda/z$

$$\begin{aligned}
f_2 &= \int_1^\lambda z^{\frac{1}{n}-1} (z^2 - 1)^{1-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n}-1} dz \\
&= \lambda^{\frac{3}{n}-2} \int_1^\lambda x^{-\frac{1}{n}-1} (x^2 - 1)^{\frac{1}{n}-1} (\lambda^2 - x^2)^{1-\frac{1}{n}} dx
\end{aligned}$$

Hence (using $x = (\lambda - 1)y + 1$)

$$f'_2(1) = 2\left(1 - \frac{1}{n}\right) \lim_{\lambda \rightarrow 1} \int_1^\lambda x^{-\frac{1}{n}-1} (x^2 - 1)^{\frac{1}{n}-1} (\lambda^2 - x^2)^{-\frac{1}{n}} dx$$

$$\begin{aligned}
&= 2\left(1 - \frac{1}{n}\right) \lim_{\lambda \rightarrow 1} \int_0^1 (\lambda - 1)((\lambda - 1)y + 1)^{-1 - \frac{1}{n}} (1 - ((\lambda - 1)y + 1)^2)^{\frac{1}{n} - 1} (\lambda^2 - ((\lambda - 1)y + 1)^2)^{-\frac{1}{n}} dy \\
&= 2\left(1 - \frac{1}{n}\right) \int_0^1 (2y)^{\frac{1}{n} - 1} (2 - 2y)^{-\frac{1}{n}} dy \\
&= \left(1 - \frac{1}{n}\right) \frac{\pi}{\sin \frac{\pi}{n}}
\end{aligned}$$

More easily,

$$f_1(1) = n$$

This implies the statement about $\text{pq}'_f(1)$. Concerning $\text{pq}'_g(1)$, one easily evaluates $g_2(1) = 0$ and $g_1(1) = n$. Then it follows that $\text{pq}'_g(1) = \frac{1}{n}g'_2(1)$. To evaluate $g'_2(1)$, we proceed as before (using $z = (\lambda - 1)y + 1$):

$$\begin{aligned}
g'_2(1) &= \frac{2}{n} \lim_{\lambda \rightarrow 1} \int_1^\lambda z^{\frac{1}{n} - 1} (z^2 - 1)^{-\frac{1}{n}} (\lambda^2 - z^2)^{\frac{1}{n} - 1} dz \\
&= \frac{1}{n} \lim_{\lambda \rightarrow 1} \int_0^1 (y)^{-\frac{1}{n}} (1 - y)^{\frac{1}{n} - 1} dy \\
&= \frac{\pi}{n \sin \frac{\pi}{n}}
\end{aligned}$$

□

Now we have the following crucial

Proposition 1 *For $n \geq 3$, The period quotient functions satisfy*

$$\begin{aligned}
\text{pq}''_f / \text{pq}'_f &> 0 \\
\text{pq}''_g / \text{pq}'_g &< 0
\end{aligned}$$

This Proposition is proven in two steps. First we deduce a linear differential equation of second order for the periods. Then we will use the important lemma 6 to compute the second derivative of the period quotients.

To begin, consider a path γ in $\overline{\mathbb{C}}$ such that $z^a(z - 1)^b(z - \lambda)^c$ does not change its value when continued analytically along γ and define

$$F_\gamma(a, b, c; \lambda) := \int_\gamma z^a(z - 1)^b(z - \lambda)^c dz$$

This function is well known to satisfy a hypergeometric differential equation

Proposition 2 F satisfies the second order differential equation

$$F'' = AF + BF'$$

with

$$\begin{aligned} A &= -(a + b + c + 1)c/(\lambda(\lambda - 1)) \\ B &= ((a + b + 2c)\lambda - a - c)/(\lambda(\lambda - 1)) \end{aligned}$$

The proof of this proposition is classical and can be found in [12].

Example 1

$$\tilde{f}_\gamma(\lambda) = \int_\gamma z^{\frac{1}{2n}-1}(1-z)^{1-\frac{1}{n}}(\lambda-z)^{\frac{1}{n}-1} dz$$

satisfies

$$\tilde{f}_\gamma'' = \frac{n-1}{2n^2\lambda(\lambda-1)}\tilde{f}_\gamma - \frac{4n-3}{2n\lambda}\tilde{f}_\gamma'$$

and

$$\tilde{g}_\gamma(\lambda) = \int_\gamma z^{\frac{1}{2n}-1}(1-z)^{-\frac{1}{n}}(\lambda-z)^{\frac{1}{n}} dz$$

satisfies

$$\tilde{g}_\gamma'' = -\frac{1}{2n^2\lambda(\lambda-1)}\tilde{g}_\gamma - \frac{2n-3}{2n\lambda}\tilde{g}_\gamma'$$

Now we transform this information into our notation:

Lemma 5 Suppose that \tilde{F} satisfies

$$\tilde{F}'' = A\tilde{F} + B\tilde{F}'$$

Then $F = \tilde{F} \circ \phi$ satisfies

$$F'' = (A \circ \phi)\phi''F + \left((B \circ \phi)\phi' + \frac{\phi''}{\phi'} \right) F' \quad (8)$$

Proof: We compute

$$\begin{aligned} F' &= (\tilde{F}' \circ \phi)\phi' \\ F'' &= (\tilde{F}'' \circ \phi)\phi'^2 + (\tilde{F}' \circ \phi)\phi'' \\ &= \left((A \circ \phi)(\tilde{F} \circ \phi) + (B \circ \phi)(\tilde{F}' \circ \phi) \right) \phi'^2 + (\tilde{F}' \circ \phi)\phi'' \\ &= (A \circ \phi)F\phi'^2 + \left((B \circ \phi)\phi' + \frac{\phi''}{\phi'} \right) F' \end{aligned}$$

□

This lemma will only be used in the following case:

Corollary 2 *If $\tilde{F}(\lambda)$ satisfies $\tilde{F}'' = A\tilde{F} + B\tilde{F}'$, then $F(\lambda) = \tilde{F}(\lambda^2)$ satisfies*

$$F''(\lambda) = 4\lambda^2 A(\lambda^2)F(\lambda) + \left(2\lambda B(\lambda^2) + \frac{1}{\lambda}\right) F'(\lambda)$$

Example 2 *Using the \tilde{f}_γ from example 1, we obtain that*

$$f_\gamma(\lambda) = \tilde{f}_\gamma(\lambda^2)$$

satisfies

$$f_\gamma'' = \frac{2(n-1)}{n^2(\lambda^2-1)} f_\gamma - 3\frac{n-1}{n\lambda} f_\gamma'$$

and that

$$g_\gamma(\lambda) = \tilde{g}_\gamma(\lambda^2)$$

satisfies

$$g_\gamma'' = -\frac{2}{n^2(\lambda^2-1)} g_\gamma - \frac{n-3}{n\lambda} g_\gamma'$$

The following lemma 6 shows how we want to use the differential equation to obtain convexity statements:

Lemma 6 *Suppose that F_1, F_2 satisfy*

$$F'' = AF + BF'$$

Then the quotient $\text{pq}_F = \frac{F_2}{F_1}$ satisfies

$$\frac{\text{pq}_F''}{\text{pq}_F'} = B - 2\frac{F_1'}{F_1}$$

Proof: The proof consists of the first step in the standard proof for the formula of the Schwarzian derivative of the quotient w , see [12]: Differentiating $F_2 = \text{pq}_F F_1$ twice gives

$$\begin{aligned} F_2' &= \text{pq}_F' F_1 + \text{pq}_F F_1' \\ F_2'' &= \text{pq}_F'' F_1 + \text{pq}_F' F_1' + \text{pq}_F F_1'' \end{aligned}$$

Using the differential equation for F_i and substituting $\text{pq}_F F_1$ for F_2 gives

$$B \cdot \text{pq}_F' F_1 = \text{pq}_F'' F_1 + \text{pq}_F' F_1'$$

and the claim follows. □

Proof: We now come to the proof of proposition 1 and theorem 5. The idea is to show the convexity properties of the period quotients by using lemma 6 and estimating the integrals explicitly.

For the first quotient, denote by $r = \text{pq}_f = f_2/f_1$. Then (using $\lambda^2 - z^2 < \lambda^2$) we obtain

$$\begin{aligned} \frac{r''}{r'} &= -3\frac{n-1}{n\lambda} - 2\frac{f_1'}{f_1} \\ &= -3\frac{n-1}{n\lambda} + 2\left(1 - \frac{1}{n}\right) 2\lambda \frac{\int_0^1 z^{\frac{1}{n}-1}(1-z^2)^{1-\frac{1}{n}}(\lambda^2-z^2)^{\frac{1}{n}-2} dz}{\int_0^1 z^{\frac{1}{n}-1}(1-z^2)^{1-\frac{1}{n}}(\lambda^2-z^2)^{\frac{1}{n}-1} dz} \\ &> \frac{n-1}{n\lambda} \\ &> 0 \end{aligned}$$

Using the asymptotic estimates (lemma 4), one obtains that r is strictly convex and increasing.

Similarly, we obtain for the second quotient $r = \text{pq}_g = g_2/g_1$. Then (using lemma 3) we get for $n \geq 3$

$$\begin{aligned} \frac{r''}{r'} &= -\frac{n-3}{n\lambda} - 2\frac{g_1' - g_2'}{g_1 - g_2} \\ &< 0 \end{aligned}$$

proving concavity of pq_g . This concludes the proof of proposition 1. For $n \geq 3$, proposition 1 together with lemma 4 imply the theorem 5 instantly. The remaining case $n = 2$ has been treated differently by the second author in [15] \square

5 Extremal Length Argument

In this section, we use an extremal length argument to reduce the two-dimensional case to the 1-dimensional case.

Recall that from the discussion in section 3, we are left with two possible candidates of Weierstraß data, represented by the pair of domains in figures 7,11

Theorem 6 *There are no surfaces of type (Ba) or (Be)*

Proof: Suppose there is a surface of type (Ba). Then there is a pair of domains $\Omega(Gdh)$ and $\Omega(\frac{1}{G}dh)$ shown as in figure 15 which are conjugate and conformal. Consider the family of curves Γ connecting the segment $(-\mu, -\lambda)$ to the segment (λ, μ) . By conformality of the domains, we have for the extremal lengths of this curve family in the respective domains

$$\text{ext}_{\Omega(Gdh)} \Gamma = \text{ext}_{\Omega(\frac{1}{G}dh)} \Gamma$$

Now we modify these domains by extending the segments $(1, \lambda)$ and $(-1, -\lambda)$ as indicated by the dashed lines. This modification is always possible for $\Omega(Gdh)$, because by conjugacy the (branched) vertex in $\Omega(Gdh)$ corresponding to ∞ and indicated by a fat dot lies above (resp. to the left of) the segment $(1, \lambda)$ (resp. $(-1, -\lambda)$).

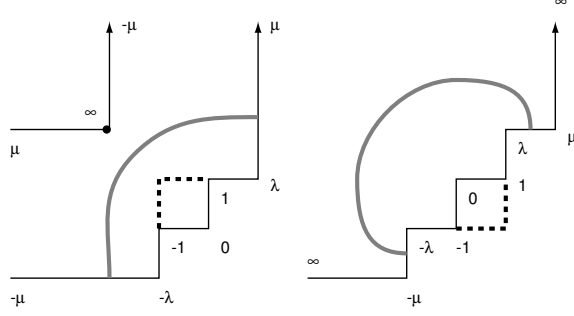


Figure 15: Extremal Length Argument

Call the domains resulting from this surgery $\Omega'(Gdh)$ and $\Omega'(\frac{1}{G}dh)$. Denote by Γ' the family of curves connecting the segment $(-\mu, -\lambda)$ with the segment (λ, μ) within these domains. By extremal length comparison (see [1], theorem 2) we have

$$\begin{aligned} \text{ext}_{\Omega'(Gdh)} \Gamma' &\geq \text{ext}_{\Omega(Gdh)} \Gamma \\ \text{ext}_{\Omega'(\frac{1}{G}dh)} \Gamma' &\leq \text{ext}_{\Omega(\frac{1}{G}dh)} \Gamma \end{aligned}$$

Now observe that the new pair of domains $\Omega'(Gdh)$ and $\Omega'(\frac{1}{G}dh)$ represent conjugate domains for the surface of type **Aa**. Then $\Omega'(\frac{1}{G}dh)$ is conformal to a domain $\Omega''(Gdh)$ of the same qualitative shape as $\Omega'(Gdh)$ so that

$$\text{ext}_{\Omega''(Gdh)} \Gamma' = \text{ext}_{\Omega'(\frac{1}{G}dh)} \Gamma'$$

Without changing the conformal type, we scale this domain so that the period of the cycle γ_1 connecting the edge $(-1, 0)$ with the edge $(1, \lambda)$ has the same length as in $\Omega'(Gdh)$. Then, by theorem 5, we know that in $\Omega''(Gdh)$ the period of the cycle γ_2 connecting the edge $(1, \lambda)$ with the edge (μ, ∞) is shorter than in $\Omega'(Gdh)$. Again by extremal length comparison,

$$\text{ext}_{\Omega''(Gdh)} \Gamma' > \text{ext}_{\Omega'(Gdh)} \Gamma'$$

Note that this time we have a strict inequality because for these (symmetric) domains, the extremal length determines the conformal structure and hence is strictly monotone in the edge data.

Putting these inequalities together gives a contradiction.

The case **Be** is treated in the very same way, using surgery to reduce the domains to domains of type **Aa** again. \square

We conclude by remarking that this theorem remains true if we consider the corresponding candidate surfaces of higher dihedral symmetry. The argument remains completely the same and relies now on the nonexistence of the higher dihedral symmetry case for surfaces of type **Aa**.

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