On Karcher's Twisted Saddle Towers

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1 Introduction

Progress is usually judged by its achievements, and this also applies to mathematics. Therefore, the simplicity of a new idea is frequently well hidden under technical ballast which is necessary to press new results out of the idea.

I feel this is true in particular for the flat structures on surfaces which Mike Wolf and myself have been using to solve high-dimensional period problems for minimal surfaces ([WW98, WW, WW01, Web99, WHW01]).

I would like to take this opportunity to apply our method to a simple class of examples which deserves more attention: Karcher's twisted saddle towers. We will see that the relatively harmless 1-dimensional period problem which one has to solve here *disappears completely* if one uses the right flat coordinate system.

Moreover, for these examples, we don't need to talk about flat structures at all, euclidean polygons will do.

The only prerequisites for this paper are familiarity with the Weierstraß representation and knowledge of the classical Schwarz-Christoffel formula.

We begin now with an informal description of the original *untwisted surfaces*. The classical singly periodic Scherk surface is invariant under a vertical translation, so it is natural to do all computations and arguments on the quotient surface. This is a sphere with four punctures which represent the four orthogonal cylindrical ends.

There is a simple variation of this: For every integer $k \geq 2$, there is a singly periodic minimal surface S_k, invariant under a vertical translation, such that the quotient surface is a sphere with $2k$ cylindrical ends. Here, consecutive ends make an angle π/k (see figure 1). S_2 is then the original Scherk surface.

One can in fact deform these surfaces by changing the angles between the ends, but this doesn't concern us here, we are interested in another deformation of this *most symmetric* case.

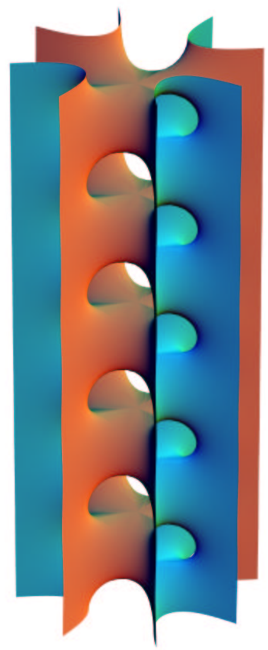

Fig. 1. Singly periodic Scherk surface S_4 with eight ends

To describe this twist deformation, fix some integer $k \geq 2$. For any *twist parameter* $0 < \phi < \pi/k$, there is an embedded singly periodic minimal surface $KS_{k,\phi}$, invariant under a vertical screw motion, whose rotational part is by angle $2 \cdot \phi$, such that the quotient surface is a sphere with $2k$ *helicoidal* ends. If we normalize the vertical translation to be of amount 1, the surfaces $KS_{k,\phi}$ converge to S_k for $\phi \to 0$.

In [Kar88], the existence of these surfaces is shown for any $k \geq 2$ and $0 < \phi < \pi/k$ in two different ways: First, using a Plateau construction with barriers, and secondly using the Weierstraß representation. While the the Plateau construction is elegant and non-computational, it uses non-elementary machinery. On the other hand, the Weierstraß representation approach requires the solution of a period problem. Neither method shows that the constructed surfaces come in a smooth family.

We will give a simple proof of this existence result, using the Weierstrass representation enriched by the usage of euclidean geometry. As a byproduct, we will obtain the smoothness of the family $KS_{k,\phi}$ in ϕ.

These surfaces are significant for at least two reasons:

For one, they were the first and still are the simplest examples of screw-motion invariant minimal surfaces. Since then, many other such surfaces have been found ([WF88, Lyn93, CHK93, HKW93, HKW99, Web00, WHW01, TW01]).

In general, such screw-motion invariant surfaces are hard to describe by the usual Weierstraß representation approach, because the Gauß map is not single valued on the quotient surface. Karcher's way to overcome this was to use the logarithmic derivative of the Gauß map, which is well-defined on the quotient, but this made the period problem a little bit more difficult as it requires an additional integration.

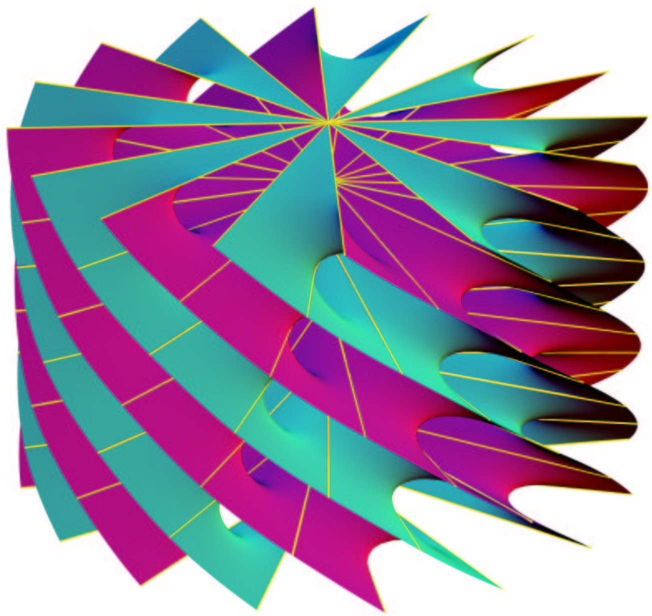

Fig. 2. A twisted Karcher-Scherk surface with eight helicoidal ends

For two, the $KS_{k,\phi}$ surfaces show a surprising phenomenon which has changed from its status as a curiosity into a focus of current research: When the twist parameter ϕ approaches its limit π/k, the minimal surface will degenerate. This degeneration can be made visible in two interesting ways by rescaling: First, one can obtain a single helicoid in k different ways, one for each pair of opposite helicoidal ends of $KS_{k,\phi}$. Secondly, one can obtain a minimal limit foliation by horizontal planes in the sense of Colding and Minicozzi.

This foliation will have k singular vertical lines through the k^{th} roots of unity. These correspond to the k collapsed helicoids from the first case of limits.

This means that close to the limit $\phi = \pi/k$, the surface will look like k copies of a helicoid glued together. Once this is observed, one can obtain many new examples by 'glueing together helicoids', appropriately positioned and in a suitable way, see [TW01].

2 The Weierstrass Representation a la Euclid

In this section, we review the Weierstraß representation for all relevant surfaces and relate it to the Schwarz-Christoffel formula.

2.1 Singly periodic Scherk surfaces with k-fold dihedral symmetry

We will discuss briefly the Weierstraß representation of the singly periodic Scherk surfaces S_k with k-fold dihedral symmetry.

We do this on the quotient surface Σ_k obtained by dividing S_k by a minimal vertical translation. This is a conformal sphere, punctured at $2k$ point at the $(2k)^{\text{th}}$ roots of unity. The points 0 and ∞ are umbilic points with vertical normal.

Furthermore, the $2k$ lines $t \mapsto te^{j\pi i/k}, \quad j = 0, \ldots, 2k-1$ are mapped to straight lines on Σ_k.

Here is the Weierstraß representation for S_k, see [Kar89, Kar88] for a detailed discussion. The choices of the signs makes the formula consistent with later formulas.

$$G(w) = -iw^{k-1} \tag{1}$$

$$dh = i\frac{w^{k-1}dw}{1-w^{2k}} \tag{2}$$

All claims made about the surfaces so far may be checked directly using these formulas. For our purposes it is important to note that the vertical translation of the surfaces is represented by a cycle around one puncture.

More concretely, a curve starting at 0, going once around one puncture and then back to 0, will be mapped to a curve connecting two umbilic points which are above each other. To check this, one has to compute period integrals involving the Weierstraß 1-forms Gdh, $\frac{1}{G}dh$ and dh. For the explicit computation, we again refer to [Kar88], but it should be clear that such a computation is simple, as periods of meromorphic 1-forms on a sphere are just residues.

2.2 The twisted Karcher-Scherk surfaces

The twisted Karcher-Scherk surfaces $KS_{k,\phi}$ are invariant under a vertical screw motion with twist angle 2ϕ. We write $\phi = a\pi$.

These surfaces exist for $0 < a < \frac{1}{k}$.

The quotient surface $\Sigma_{k,\phi}$ is again a $2k$-punctured sphere. A slight complication arises because it is not possible to have the punctures at the roots of unity. Using an additional parameter r, one places the punctures at the roots of $w^k = r^k$ and $w^k = -1/r^k$.

The straight lines of S_k survive the deformation as straight lines, this is what made the Plateau construction possible.

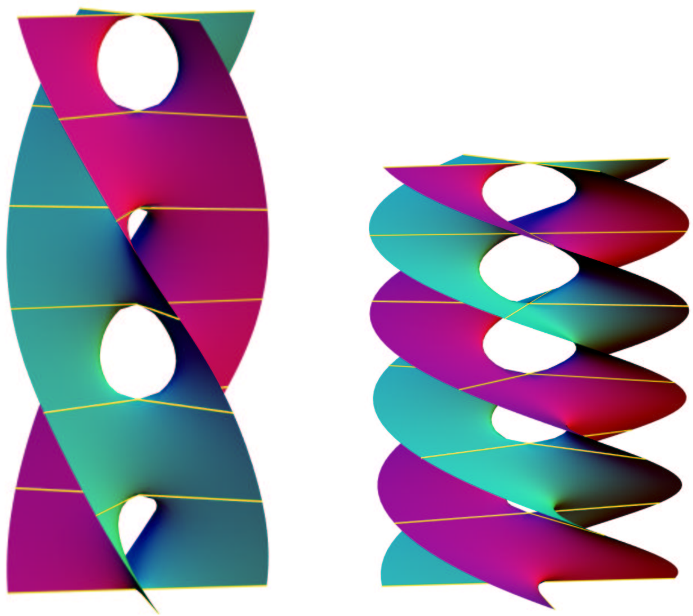

Fig. 3. Twisted Karcher-Scherk surface with four helicoidal ends

The real parameter r is to be determined so that a horizontal period condition is satisfied. This again requires the evaluation of period integrals. While the height differential is still quite simple, the Gauss map is much more complicated then for S_k. Here is the formula from [Kar88]:

$$G(w) = -iw^{k-1} \cdot \left(\frac{r^k - w^k}{r^k w^k + 1} \right)^a \tag{3}$$

$$dh = \frac{iw^{k-1}}{(r^k - w^k)(r^k w^k + 1)}\, dw \tag{4}$$

For $a = 0$ and $r = 1$, we obtain the classical Scherk surfaces S_k, see (1).

The Gauß map has the unpleasant feature that it is not anymore single valued on the quotient surface.

In fact, the multi-valuedness measures precisely the twist angle: If we analytically continue $G(z)$ around a puncture, the value $G(z)$ changes to $e^{a\cdot 2\pi i}$, so that $2\phi = 2\pi a$ is the twist angle.

2.3 Towards the Schwarz-Christoffel formula

The first step in simplifying the Weierstraß representation is to exploit the k-fold symmetries. This is done by a simple coordinate change:

Write $z = w^k$ and $R = r^k$. Then, up to constant real factors,

$$\begin{aligned}
G(z) &= -iz^{1-1/k} \cdot \left(\frac{R-z}{Rz+1}\right)^a \\
dh &= \frac{idz}{(R-z)(Rz+1)} \\
Gdh &= z^{1-\frac{1}{k}}(R-z)^{a-1}(Rz+1)^{-a-1}\, dz \\
\frac{1}{G}dh &= -z^{\frac{1}{k}-1}(R-z)^{-a-1}(Rz+1)^{a-1}\, dz
\end{aligned}$$

The key observation now is that the 1-forms Gdh and $\frac{1}{G}dh$ are single-valued in the upper half plane and represent integrands in the Schwarz-Christoffel formula for the Riemann map from the upper half plane to euclidean polygons. Let us pause for a minute to review this formula.

Definition 1. *A* euclidean polygon *is a simply connected region in the plane such that all boundary components consist of straight lines and segments, meeting in a discrete set of vertices.*

That is, we allow for unbounded polygons but exclude domains with holes.

By the Riemann mapping theorem, there is a biholomorphic map f from the upper half plane to any such euclidean polygon. This map extends continously and one-to-one to the (extended) real axes. Let us denote the preimages of the vertices by t_i, this will be a finite subset of the (extended) real line.

Denote by $\alpha_j = (a_j - 1)\pi$ the interior angle of the polygon at $f(t_j)$. Then we have

Theorem 1 (Schwarz-Christoffel formula). *Given a euclidean polygon as above, there are constants C_1, C_2 and w_0 such that*

$$f(z) = C_1 + C_2 \int_{w_0}^{w} \prod_j (z - t_j)^{a_j}\, dz$$

Now back to Gdh and $\frac{1}{G}dh$.

We conclude from theorem 1 that the map $z \mapsto \int^z Gdh$ maps the upper half plane to an open euclidean quadrilateral such that the points $0, R, -\frac{1}{R}, \infty$ are mapped to vertices with angles $(2-\frac{1}{k})\pi, a\pi, -a\pi, \frac{1}{k}\pi$. Similarly, the map $z \mapsto \int^z Gdh$ maps the upper half plane to an open euclidean quadrilateral such that the points $0, R, -\frac{1}{R}, \infty$ are mapped to vertices with angles $\frac{1}{k}\pi, -a\pi, a\pi, (2-\frac{1}{k})\pi$.

2.4 The period condition

This is a quite remarkably simple interpretation of Gdh and $\frac{1}{G}dh$, but what does it help? We still have to determine the parameter $R = r^k$ in the Weierstraß representation such that a certain period condition is satisfied. We will now make this condition explicit:

Consider two closed cycles: γ_1 commencing at 0, and looping around R back to 0, and γ_2 commencing at 0 and around $-1/R$ in the z-plane. These will be mapped to curves on the surface which represent a purely vertical translation. The verticality is taken care of by the definition of dh, but we haven't argued why the two endpoints of image curves lie strictly above each other. This is the content the *horizontal period condition* which can be stated as

$$\int_{\gamma_i} Gdh = \overline{\int_{\gamma_i} \frac{1}{G}dh} \tag{5}$$

But, in terms of our understanding of Gdh and $\frac{1}{G}dh$ as integrands of Schwarz-Christoffel integrals, this boils down to a pair of geometric conditions on the euclidean image quadrilaterals. We will exhibit this geometric condition in the next section.

Moreover, if we can find a pair of image quadrilaterals which are conformally equivalent (as they must both be conformal to the upper half plane quadrilateral with vertices at $0, R, \infty, -\frac{1}{R}$), have the correct angles, and satisfy the geometric conditions, we can *define* Gdh and $\frac{1}{G}dh$ as Schwarz-Christoffel integrands for these quadrilaterals, without using any formulas. This is our next goal, so we will forget about minimal surfaces and the Weierstraß representation for a little while.

3 A geometric construction of the Weierstraß data

We begin with an elementary construction:

For $0 < a < \frac{1}{k}$, construct an acute triangle with angles $\frac{1}{k}\pi$, $b\pi = (\frac{1}{2} - \frac{1}{2k} - \frac{a}{2})\pi$ and $c\pi = (\frac{1}{2} - \frac{1}{2k} + \frac{a}{2})\pi$. In honor of the case $k = 2$, we call the edge opposite to the vertex with angle $\frac{1}{k}\pi$ the *hypothenuse* of the triangle.

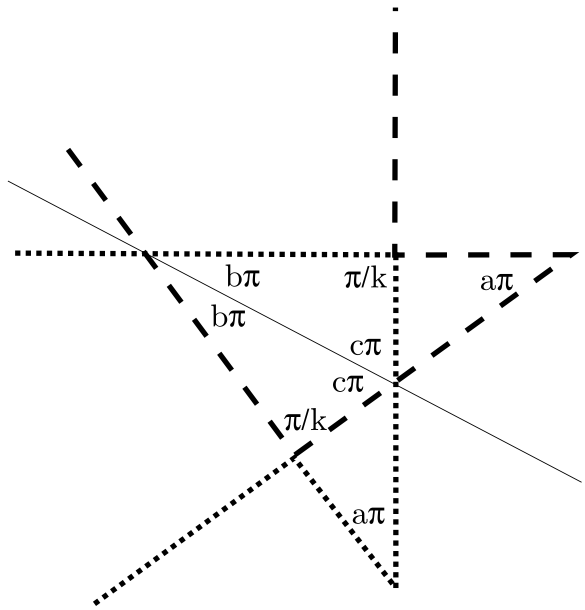

Fig. 4. Construction of the quadrilaterals

Observe that this construction is possible if and only if $0 < a < \frac{1}{k}$, because we will need the initial triangle to be acute.

Remark 1. To be precise, the construction is also possible for all a with $0 < a < \frac{1}{k}$ modulo 2π. This leads to *branched* polygons and *overtwisted* Karcher-Scherk surfaces which are not embedded any longer.

Reflect the triangle at its hypothenuse and extend all edges to infinity. Mark the edges as indicated as *dotted* and *dashed*. Observe that the angle $a\pi$ in the figure is implied by the construction.

Separate the dashed and dotted polygons, and reflect the dashed polygon at the x-axes. These polygons form now two unbounded quadrilaterals Q_1 and Q_2 in two different euclidean planes. Label the vertices (including the vertex at infinity) with the symbols $0, R, -1/R, \infty$ as indicated.

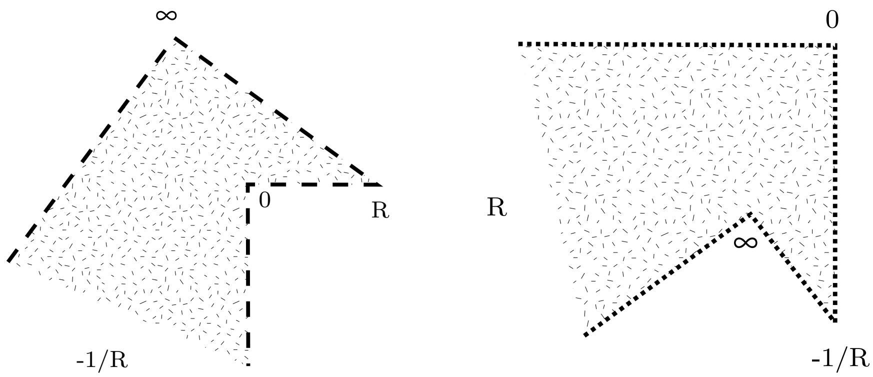

Fig. 5. Separation in dashed and dotted domains Q_1 and Q_2

Define maps f_1 and f_2 from the upper half plane to the dashed and dotted domains Q_1 and Q_2 so that labels are mapped to labels. This *defines* uniquely the real value of R as the location of the symbol on the real axes.

That the same R works for both domains follows from the fact that both domains are congruent and have the same modulus. We will see this shortly also by a simple computation.

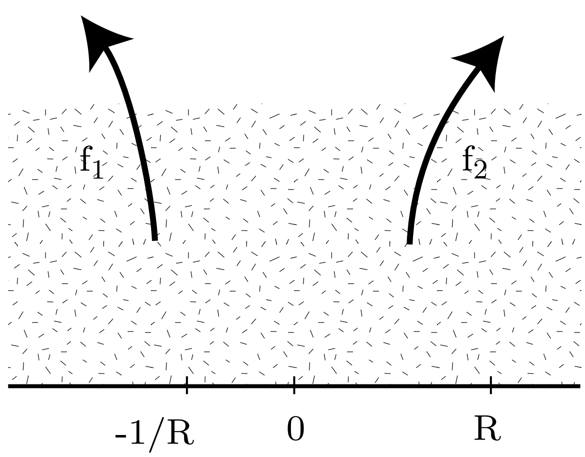

Fig. 6. The Schwarz-Christoffel maps

If we now *define*

$$\begin{aligned} G dh &= df_1(z) = f_1'(z)dz \\ \frac{1}{G} dh &= df_2(z) = f_2'(z)dz \end{aligned}$$

then the Schwarz-Christoffel formula implies that, up to rotation and scaling,

$$\begin{aligned} Gdh &= z^{1-\frac{1}{k}}(R-z)^{a-1}(Rz+1)^{-a-1} \\ \frac{1}{G}dh &= -z^{\frac{1}{k}-1}(R-z)^{-a-1}(Rz+1)^{a-1} \end{aligned}$$

This is exactly the kind of formula we were aiming for.

To verify that these formulas are correct without rotating or scaling, we need to check that the image domains Q_1 and Q_2 under the Schwarz-Christoffel maps $\int Gdh$ and $\int 1/Gdh$ are *congruent* and *correctly aligned* as in the figure.

For the correct alignment, observe that the integrand Gdh (resp. $\frac{1}{G}dh$) is positive (resp. negative) in the interval $(0, R)$, so that this interval is mapped to horizontal lines with the correct orientation as in the figure.

For the congruence, we use the automorphism $\iota(z) = -\frac{1}{z}$ of the upper half plane (which permutes the vertices $0, R, \infty$ and $-1/R$) and compute that

$$\begin{aligned} \iota^* Gdh &= \left(\frac{1}{z}\right)^{1-\frac{1}{k}} \left(R+\frac{1}{z}\right)^{a-1} \cdot \left(1-\frac{R}{z}\right)^{-1-a} \frac{dz}{z^2} \\ &= e^{i\theta}\, \frac{1}{G}dh \end{aligned}$$

for some real number θ. This means that Q_1 and Q_2 differ by a rotation (and not by a scale factor).

We now verify that the horizontal period conditions are satisfied. Let's begin with the condition on the cycle γ_1 which encircles R. The condition requires to compute the periods $\int_{\gamma_1} Gdh$ and $\int_{\gamma_1} \frac{1}{G}dh$. The cycle γ_1 commences at 0, runs in the upper half plane to some point on the segment (R, ∞), and then continues back (say by reflection at the real axes) in the lower half plane to 0. Thus, the vector $u_1 = \int_{\gamma_1} Gdh$ can be made visible in the image quadrilateral Q_1 as the vector from 0 to the mirror image of 0 at the straight segment $[R, \infty]$. The same statement holds for $v_1 = \int_{\gamma_1} \frac{1}{G}dh$ in Q_2.

Similarly, recall that γ_2 is a cycle commencing at 0, crossing the segment $(-\infty, -\frac{1}{R})$ and going back to 0. Thus the period vector $u_2 = \int_{\gamma_2} Gdh$ (resp. $v_2 = \int_{\gamma_2} \frac{1}{G}dh$) is the vector from 0 to the mirror image of 0 at the edge $[-\infty, -\frac{1}{R}]$.

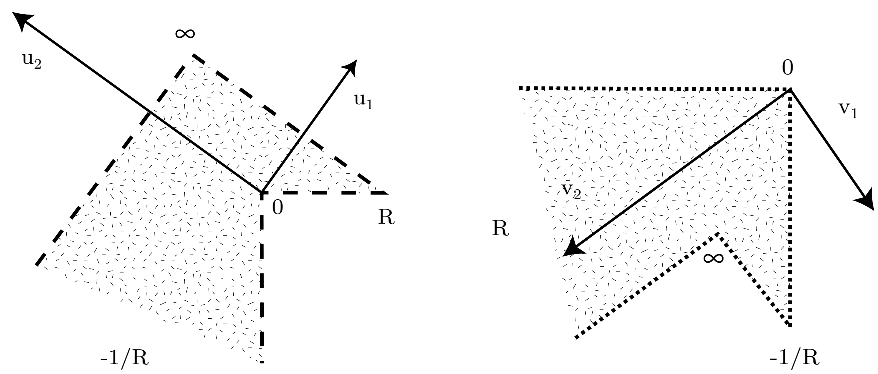

Fig. 7. Conjugacy of the period vectors

By the geometric construction of the quadrilaterals, the period vectors U_1, v_1 and u_2, v_2 are complex conjugate. This shows that that the period condition (5) is satisfied.

This finishes the construction of the twisted Karcher-Scherk surface for any $k \geq 2$ and $0 < \phi < \pi/k$. As the domains Q_1 and Q_2 vary smoothly with ϕ, so do the the 1-forms Gdh and $\frac{1}{G}dh$, and we obtain that the family of surfaces is smooth in ϕ as a byproduct.

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