PERIOD QUOTIENT MAPS OF MEROMORPHIC 1-FORMS AND MINIMAL SURFACES ON TORI

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Abstract. Consider on a complex 1-dimensional torus T_λ an abelian differential of the second kind α_λ. Assign to each λ the period quotient of α_λ for two independent cycles on T_λ. For appropriate choices of α_λ, the image of the modular sphere under this map is simple to describe, extending the classical case of holomorphic α_λ. Using this information for different α_λ simultaneously, we discuss applications to existence and uniqueness questions for the Chen-gackstatter surface, the Costa surface, and the translation invariant periodic helicoid with handles. In particular, there is now a conceptual and essentially non-computational proof for the uniqueness of the Chen-Gackstatter surface.

1. Introduction

In this paper, we study the geometric properties of maps from the modular sphere of elliptic curves to the Riemann sphere $\hat{\mathbb{C}}$ given by quotients of the periods of meromorphic 1-forms.

The holomorphic case is quite classical:

For $\lambda \in \hat{\mathbb{C}} - \{0, 1, \infty\}$, denote by

$$y^2 = x(x-1)(x-\lambda)$$

an elliptic curve T_λ, with modular invariant λ. Then

$$\omega = \frac{dx}{y}$$

is a holomorphic 1-form on T_λ without zeros. A period of ω is its integral over a cycle $\alpha \in H_1(T_\lambda, \mathbb{Z})$:

$$\omega_\alpha(\lambda) = \int_\alpha \omega,$$

and for a suitable choice of cycles γ_1, γ_2, the period quotient

$$\tau = \tau(\lambda) = \omega_{\gamma_2}(\lambda)/\omega_{\gamma_1}(\lambda)$$

is the lattice constant τ of the torus.

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As the cycles γ_1, γ_2 cannot be chosen independent of λ, both the periods and the lattice constant are naturally multivalued functions of λ, the multivaluedness becomming apparent for λ varying in a neighborhood of $0, 1$, or ∞.

As one of our principal goals will be to compare the images of period quotient maps for the form ω as well as for other meromorphic 1-forms, we need to make all these choices consistently. This is done as follows:

First, we restrict our attention to $\lambda = -1$ (the square torus). The cycles γ_1 and γ_2 are chosen for this λ as follows: Let γ_1 to be a simple closed curve in $\hat{\mathbb{C}} - \{0, 1, -1, \infty\}$ with winding number 1 around 0 and -1, but winding number 0 around 1 and ∞.

Similarly, let γ_2 be a simple closed curve in $\hat{\mathbb{C}} - \{0, 1, -1, \infty\}$ with winding number 1 around 0 and 1, but winding number 0 around -1 and ∞.

Each of these curves has two different closed lifts γ_1 and γ_2 to T_λ which are homologous up to orientation.

The orientation of the lifts are chosen now so that (still for $\lambda = -1$) $\int_{\gamma_1} \omega \in \mathbb{R}^+$ and $-i \int_{\gamma_2} \omega \in \mathbb{R}^+$. This is possible, because the integrand $dx/y = 1/\sqrt{x(x^2-1)}$ is purely real (resp. purely imaginary) on the segment $(-1, 0)$ (resp. $(0, 1)$).

Of course we think of γ_i as representing the standard horizontal and vertical cycles on a lattice torus. Now that we have defined $\omega_1(\lambda)$ and $\omega_2(\lambda)$ uniquely for $\lambda = -1$, we extend this to λ close to -1, using the same cycles γ_1 and γ_2 and the same topological choice of lifts. This defines the periods as holomorphic functions in λ in a neighborhood of -1.

For other λ, these periods are defined by analytic continuation. This would give multivalued functions in $\hat{\mathbb{C}} - \{0, 1, \infty\}$. However we are content to extend the periods as single-valued functions just to the upper half plane.

Now define

$$\omega_i = \int_{\gamma_i} \omega$$

Denote the period quotient by $\tau = \omega_2/\omega_1$. Our convention implies that $\tau \in \mathbb{H}$ (the upper half plane). Let Λ denote the lattice spanned by ω_1 and ω_2.

The map

$$T_\lambda \to \mathbb{C}/\Lambda, \qquad p \mapsto \int^p \omega$$

maps the torus T_λ biholomorphically to the lattice torus defined by the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Using $z = \int^p \omega$ as a new local coordinate we can write $\omega = dz$.

In this coordinate z, the functions x and y on T_λ become elliptic functions $\wp(z)$ and $\wp'(z)$ in the complex plane, doubly periodic with respect to the lattice Λ spanned by ω_1 and ω_2. The meromorphic function $\wp$ has a double order pole in $z = 0$ and a double order zero in $z = (\omega_1 + \omega_2)/2$.

The following famous result is at the core of our investigations:

Theorem 1.1. *The map $\lambda \mapsto \tau = \omega_2/\omega_1$ maps the upper half plane to a circular triangle in $\mathbb{H}$ with vertices at $0, 1$ and ∞ and zero-degree angles.*

These days, this theorem is usually formulated (and proven) for the inverse map, which maps the lattice constant τ to the modular invariant λ. This map is single valued and represents a universal covering map from the upper half plane to the thrice-punctured sphere.

For our purposes, however, it is key to understand the original map $\lambda \mapsto \tau$. In this case, the theorem is proven by invoking the Schwarz-Christoffel formula for circular triangles. More precisely, one proceeds as follows:

(1) The periods $\omega_i(\lambda)$ satisfy an explicit second order linear differential equation.
(2) This implies that the Schwarzian derivative of the period quotient map $\tau(\lambda)$ is explicitly computable as well.
(3) The Riemann mapping function from the upper half plane to a circular triangle with zero-degree angles has the same Schwarzian derivative.
(4) Hence the period quotient map and the Riemann mapping function coincide up to a Möbius transformation.
(5) Explicit evaluation of the period quotient function at the points $0, 1, \infty$ show that these points are mapped to $0, 1, \infty$. As circular triangles are uniquely determined by the position of their vertices and their angles, Theorem 1.1 follows.

Our first goal (Section 2) is to generalize this theorem to the period quotient map defined by certain other meromorphic 1-forms such as $x\frac{dx}{y}$ on T_λ. The proof of this follows closely the abobe outline. The particular cases of interest to us have never been considered before. And more importantly, the results obtained have several immediate applications to existence and uniqueness results for minimal surfaces (Section 3). Finally, in Section 4 we are able to extend the analysis of the geometric behaviour to an even more general case where the image domains are circular quadrilaterals. This is important because in general, such maps cannot be determined explicitly due to the accessory parameters in the Schwarz-Christoffel formulas for circular polygons.

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Since the first draft of this paper has been distributed, F. Lopéz has called my extension to Kusner's uniqueness proof of the Chen-Gackstatter surface in the first case. Also, using the insights from the period quotient maps, F. Martín and the author ([[]MW]) have been able to characterize the Cost-Hoffman-Meeks surfaces. In addition, the singly periodic helicoid and the genus one helicoid are now much better understood ([HKW1,HKW2,HKW3,W8,WHW]), using ideas which originated in proposition 3.6.

I would also like to thank D. Hoffman for many discussions and suggestions which shaped the final form of this paper considerably.

2. The geometry of period quotient maps

In this section, we collect some basic facts and then prove the first main result, which gives a complete geometric characterization of period quotient maps for a certain class of meromorphic 1-forms.

2.1 Elliptic Functions.

Recall that we have defined the elliptic curve T_λ by the equation

$$y^2 = x(x-1)(x-\lambda) \tag{2.1}$$

Then $\omega = \frac{dx}{y}$ is a holomorphic 1-form on T_λ without zeros, and we can introduce local coordinates (well-defined up to additive constants) z on T_λ by

$$z(p) = \int^p \omega$$

The function x on t_λ, considered as a function of z will be denoted by $\wp(z)$. This is *not* the classical Weierstraß $\wp$-function but related to it by a linear transformation. Because of $\frac{dx}{dz} = y$ the algebraic equation (2.1) becomes a differential equation for $\wp$:

$$\wp'(z)^2 = \wp(z)(\wp(z)-1)(\wp(z)-\lambda) \tag{2.2}$$

In the same manner as in the introduction, we can define for $\lambda \in \mathbb{H}$ the periods of $x\frac{dx}{y}$ as

$$\psi_i = \int_{\gamma_i} x\frac{dx}{y}$$

and the period quotient

$$\eta = \psi_1/\psi_2$$

as single valued functions. We begin with the following elementary properties of $\wp, \omega$ and ψ:

Lemma 2.1. *The function $\wp(z)$ is an even elliptic function.*

$$\wp'' = \frac{1}{2}\left(3\wp^2 - 2(\lambda+1)\wp + \lambda\right) \tag{2.3}$$

$$\wp(z) = \frac{4}{z^2} + \frac{1+\lambda}{3} + \frac{1-\lambda+\lambda^2}{60}z^2 + \cdots \tag{2.4}$$

(the coefficients are polynomials in λ over $\mathbb{R}$)

$$\wp(z+\frac{\omega_1}{2}) = \frac{\wp(z)-\lambda}{\wp(z)-1} \tag{2.5a}$$

$$\wp(z+\frac{\omega_2}{2}) = \lambda\frac{\wp(z)-1}{\wp(z)-\lambda} \tag{2.5b}$$

(2.5c)

$$\wp(z+\frac{\omega_1+\omega_2}{2}) = \frac{\lambda}{\wp(z)}$$

$$\wp(u+v) = -\wp(u) - \wp(v) + \lambda + 1 + \left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2$$

$$\omega_1\psi_2 - \omega_2\psi_1 = 8\pi i \tag{2.6}$$

Proof. The first equation follows from the differential equation (2.2), the second by solving it for a Laurent series with a second order pole. The next four come from divisor considerations. The last equation (2.6) is the classical Legendre relation using our normalization. We include the proof of the Legendre relation:

Consider a fundamental parallelogram R of the lattice spanned by ω_1, ω_2 which contains the double order pole of $\wp(z)$ in its interior. We calculate

$$\begin{aligned}\int_R z\cdot\wp dz &= \int_0^{\omega_1} z\cdot\wp dz + \int_0^{\omega_2}(z+\omega_1)\wp dz + \int_{\omega_1}^0 (z+\omega_2)\wp dz + \int_{\omega_2}^0 z\cdot\wp dz\\ &= \int_0^{\omega_1}(-\omega_2)\wp dz + \int_0^{\omega_2}\omega_1\wp dz\\ &= -\omega_2\wp_1 + \omega_1\wp_2\end{aligned}$$

On the other hand by the residue theorem

$$\begin{aligned}\int_R z\cdot\wp dz &= 2\pi i\cdot \operatorname{res}_{z=0}(z\cdot\wp dz)\\ &= 8\pi i\end{aligned}$$

where we used the power series expansion of $\wp$. □

Observe that the behavior of $\wp$ under translation by half-periods is simpler than the analogous behavior of the classical Weierstraß $\wp$-function.

Corollary 2.2. *For $\lambda \in (-1,1)$, $\wp$ is real on the lines $\mathbb{R}$ and $i\mathbb{R}$ and their translates by half-periods.*

Proof. This follows for $\mathbb{R}$ and $i\mathbb{R}$ from the power series expansion and for the translates by half-periods from the transformation formulas in Lemma 2.1. In fact, we can say more, because for $\lambda \in (-1,1)$ we have a rectangular torus and here $\wp$ maps the sub-rectangle with corners at the 2-division points conformally to the upper half plane. □

Now we introduce the meromorphic 1-forms whose period quotients have the most simple non-trivial mapping behavior:

Define the meromorphic 1-form

$$\begin{aligned}\alpha(r,s;\lambda) &= x^s(x-1)^s(x-\lambda)^s\frac{dx}{y}\\ &= \wp^r(\wp-1)^s(\wp-\lambda)^s dz\end{aligned} \tag{2.7}$$

on T_λ and denote the multi-valued periods by

$$\alpha_i(r,s;\lambda) = \int_{\gamma_i}\alpha(r,s;\lambda) \tag{2.8}$$

and the multi-valued period quotient map by

$$\sigma(r,s;\lambda) = \frac{\int_{\gamma_2}\alpha(r,s;\lambda)}{\int_{\gamma_1}\alpha(r,s;\lambda)} \quad . \tag{2.9}$$

By our convention of chosing the cycles and their lifts, all these functions are well-defined and single-valued in the upper half plane.

Note that $\omega = \alpha(0,0)$ and $\psi = \alpha(1,0)$.

The period integrals are of course related to hypergeometric functions, which are well documented in the literature. However, we decided to provide complete proofs for the facts needed here, as hypergeometric functions are differently (and not consistently) normalized, and the passage from our conventions to a classical normalization would be about as tedious.

We will now follow the outline given in the introduction for the proof of the classical mapping theorem in the general case.

2.2 Differential equation for the period functions.

The first step is to prove that the periods satisfy a linear second order differential equation. For this purpose, it is helpful to consider slightly more general forms

$$\alpha(r,s,t;\lambda) = \frac{1}{y}x^r(x-1)^s(x-\lambda)^t dx$$

as this will show the dependence of the triangle angles on the parameters r, s, t more clearly. We will first need a formula for the derivatives of α with respect to λ:

Lemma 2.3. *Denote the derivative of α with respect to λ by α'. Then*

$$\alpha'(r,s,t) = \frac{1-2t}{2}\alpha(r,s,t-1) \tag{2.10}$$

Proof. From (2.1) we have

$$2yy' = -x(x-1)$$

and then we compute

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\alpha(r,s,t) &= \frac{\partial}{\partial\lambda}\frac{1}{y}x^r(x-1)^s(x-\lambda)^t dx \\
&= \frac{-tyx^r(x-1)^s(x-\lambda)^{t-1} - y'x^r(x-1)^s(x-\lambda)^t}{y^2}dx \\
&= \frac{(-2ty^2+y^2)x^r(x-1)^s(x-\lambda)^{t-1}}{2y^3}dx \\
&= \frac{1-2t}{2}\alpha(r,s,t-1)
\end{aligned}$$

□

We use (2.10) to derive a linear differential equation of second order for the periods of α over arbitrary cycles. That this must be possible is due to the fact that the three 1-forms α, α' and α'' provide linear functionals from the two-dimensional first homology $H_1(T_\lambda), \mathbb{C})$ to the complex numbers. These functionals must be linearly dependent, so a non-trivial linear combination of the forms must be exact. We now determine this linear combination explicitly:

Definition 2.4. Let γ be a cycle on T_λ. Define

$$\alpha_\gamma(r,s,t;\lambda) := \int_\gamma \alpha(r,s,t;\lambda)$$

Lemma 2.5. *We have*

$$A\alpha_\gamma + B\alpha'_\gamma + C\alpha''_\gamma = 0$$

with

$$\begin{aligned}
A &= \left(r+s+t-\frac{1}{2}\right)\left(t-\frac{1}{2}\right) \\
B &= -(r+s+2t-2)\lambda + r+t-1 \\
C &= \lambda(\lambda-1)
\end{aligned}$$

Proof. The idea is to find a meromorphic function f on T_λ such that

$$df = A\alpha + B\alpha' + C\alpha''$$

Then the claim follows by integration. Observe that the exterior derivative on the left hand side is on the torus T_λ, using the coordinate x, while the derivatives α'_γ on the right hand side are on the modular sphere, using λ. In fact, such a function is given by

$$f = \frac{1}{y} x^{r+1}(x-1)^{s+1}(x-\lambda)^{t-1}$$

To see this, first observe from (2.1) that

$$2y dy = \left(3x^2 - 2(\lambda+1)x + \lambda\right) dx$$

which gives

$$\begin{aligned} df =&(r+1)(x-1)(x-\lambda)\alpha(r,s,t-2) \\ &+ (s+1)x(x-\lambda)\alpha(r,s,t-2) \\ &+ (t-1)x(x-1)\alpha(r,s,t-2) \\ &- \frac{1}{2}\left(3x^2 - 2(\lambda+1)x + \lambda\right)\alpha(r,s,t-2) \end{aligned}$$

On the other hand, using Lemma 2.3

$$A\alpha + B\alpha' + C\alpha'' = \left(A(x-\lambda)^2 + B(x-\lambda)\frac{1-2t}{2} + C\frac{1-2t}{2}\frac{3-2t}{2}\right)\alpha(r,s,t-2)$$

Comparing the coefficients of the powers of x proves the claim. □

2.3 The Schwarzian derivative of the period quotient.

In the second step, we compute the Schwarzian derivative of the period quotient

$$\sigma(\lambda) = \sigma(r,s,t;\lambda) = \frac{\alpha_{\gamma_2}(r,s,t;\lambda)}{\alpha_{\gamma_1}(r,s,t\lambda)}.$$

Recall that the Schwarzian derivative of a non-constant meromorphic function $\sigma(\lambda)$ is defined as

$$\mathcal{S}_\lambda\sigma = \left(\frac{\sigma''}{\sigma'}\right)' - \frac{1}{2}\left(\frac{\sigma''}{\sigma'}\right)^2 \tag{2.11}$$

One of its key properties is that it is invariant by postcomposition with Möbius transformations. That is, whenever ϕ is a Möbiustransformation, we have

$$\mathcal{S}_\lambda\phi(\sigma(\lambda)) = \mathcal{S}_\lambda\sigma(\lambda)$$

For two linearly independent solutions of a linear differential equation of second order this has the following consequence: As any pair of such solutions is relate to any other pair by a linear transformation, the respective quotients are related by a Möbius transformation. Hence the Schwarzian derivative of the quotient is independent on the particular choice of the solutions, hence is determined already by the coefficients of the differential equation. Again, we want these data explicitly. Here is how:

Lemma 2.6. *Suppose $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ are linearly independent solutions of the second order differential equation*

$$A\alpha_\gamma + B\alpha_\gamma' + C\alpha_\gamma'' = 0$$

Then the Schwarzian derivative of $\sigma(\lambda) = \alpha_2(\lambda)/\alpha_1(\lambda)$ satisfies

$$\mathcal{S}_\lambda \sigma = -\left(\frac{B}{C}\right)' - \frac{1}{2}\left(\frac{B}{C}\right)^2 + 2\frac{A}{C} \tag{2.12}$$

Proof. We differentiate the equation $\alpha_2 = \sigma\alpha_1$ twice with respect to λ:

$$\begin{aligned}\alpha_2' &= \sigma'\alpha_1 + \sigma\alpha_1' \\ \alpha_2'' &= \sigma''\alpha_1 + 2\sigma'\alpha_1' + \sigma\alpha_1''\end{aligned}$$

Using the differential equation gives

$$-\frac{A}{C}\alpha_2 - \frac{B}{C}\alpha_2' = \alpha_2'' = \sigma''\alpha_1 + 2\sigma'\alpha_1' - \sigma\frac{A}{C}\alpha_1 - \sigma\frac{B}{C}\alpha_1'$$

Substituting α_2 for $\sigma\alpha_1$ gives

$$-\frac{A}{C}\alpha_2 - \frac{B}{C}\alpha_2' = \sigma''\alpha_1 + 2\sigma'\alpha_1' - \frac{A}{C}\alpha_2 - \sigma\frac{B}{C}\alpha_1'$$

and using the expression for α_2'

$$-\frac{B}{C}(\sigma'\alpha_1 + \sigma\alpha_1') = \sigma'' s_2 + 2\sigma' s_2' - \sigma\frac{B}{C}\alpha_1'$$

which simplifies to

$$-\sigma'\frac{B}{C}s_1 = \sigma''\alpha_1 + 2\sigma' s_1'$$

which implies

$$\frac{\sigma''}{\sigma'} = -\frac{B}{C} - 2\frac{\alpha_1'}{\alpha_1}$$

Using this, we compute

$$\begin{aligned}\mathcal{S}_\lambda \sigma &= \left(\frac{\sigma''}{\sigma'}\right)' - \frac{1}{2}\left(\frac{\sigma''}{\sigma'}\right)^2 \\ &= -\left(\frac{B}{C} + 2\frac{\alpha_1'}{\alpha_1}\right)' - \frac{1}{2}\left(\frac{B}{C} + 2\frac{\alpha_1'}{\alpha_1}\right)^2 \\ &= -\left(\frac{B}{C}\right)' - \frac{1}{2}\left(\frac{B}{C}\right)^2 - 2\frac{\alpha_1\alpha_1'' - (\alpha_1')^2}{\alpha_1^2} - 2\left(\frac{B}{C}\frac{a_1'}{a_1}\right) - 2\left(\frac{\alpha_1'}{\alpha_1}\right)^2 \\ &= -\left(\frac{B}{C}\right)' - \frac{1}{2}\left(\frac{B}{C}\right)^2 + 2\frac{A\alpha_1 + B\alpha_1'}{C\alpha_1} - 2\left(\frac{B}{C}\frac{\alpha_1'}{\alpha_1}\right) \\ &= -\left(\frac{B}{C}\right)' - \frac{1}{2}\left(\frac{B}{C}\right)^2 + 2\frac{A}{C}\end{aligned}$$

□

This will now be applied to the differential equation which is satisfied by the periods α_γ of $\alpha(r,s,t)$:

Corollary 2.7. *Consider two linearly independent periods $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ of the form $\alpha(r,s,t)$. Then the quotient $\sigma(\lambda) = \frac{\alpha_2(\lambda)}{\alpha_1(\lambda)}$ has the Schwarzian derivative*

$$\mathcal{S}_\lambda \sigma = \frac{1-(r+t)^2}{2\lambda^2} + \frac{1-(s+t)^2}{2(\lambda-1)^2} + \frac{t(r+s+t)-rs-\frac{1}{2}}{\lambda(\lambda-1)} \tag{2.13}$$

Proof. We apply Lemma 2.5 to Lemma 2.6:

$$\begin{aligned}
\frac{A}{C} &= \frac{(r+s+t-\frac{1}{2})(t-\frac{1}{2})}{\lambda(\lambda-1)} \\
\frac{B}{C} &= \frac{-(r+s+2t-2)\lambda + a + c - 1}{\lambda(\lambda-1)} \\
&= -\frac{r+t-1}{\lambda} - \frac{s+t-1}{\lambda-1} \\
\mathcal{S}_\lambda \sigma &= \frac{1-(r+t)^2}{2\lambda^2} + \frac{1-(s+t)^2}{2(\lambda-1)^2} + \frac{t(r+s+t)-rs-\frac{1}{2}}{\lambda(\lambda-1)} \quad \square
\end{aligned}$$

Remark 2.8. The above expression is the unique rational function on the Riemann sphere which has poles of second only at $0, 1$ and ∞ with coefficients $(1-(r+t)^2)/2, (1-(s+t)^2)/2$ and $(1-(r+s)^2)/2$ of the highest (negative) order term, respectively.

Theorem 2.9. *The period quotient map $\lambda \mapsto \sigma(r,s,t;\lambda)$ maps the upper half plane to a circular triangle with angles $|r+t|\pi, |s+t|\pi, |r+s|\pi$ at the images of the points $0, 1, \infty$*

Proof. This is an application of the Schwarz-Christoffel theorem for maps from the upper half plane to circular polygons (see [Car, Neh]): Take a triangle bounded by circular arcs with angles $a_i\pi$ as given in the theorem. Such a triangle is unique up to Möbius transformations. By the Riemann mapping theorem, there is unique holomorphic map from the upper half plane to a circular triangle, normalized so that the points $0, 1, \infty$ are mapped to the vertices. Extend this map continuously to the real axes and continue it analytically by the Schwarz reflection principle to the lower half plane. Even though this map is not single valued, its Schwarzian derivative is (again due to the fact that it is invariant under postcomposition by Möbius transformations). The angles $|a_i|\pi$ at the image vertices of $p_i \in \{0, 1, \infty\}$ of the triangle determine the coefficients of the poles of second order of the Schwarzian derivative by the Schwarz-Christoffel formula to be $(1-a_i^2)/2$ (this follows from a local analysis near $0, 1$ and ∞ and Liouville's theorem). This proves that the Schwarzian derivative (2.13 of our period quotient map is the same as the Schwarzian derivative of the triangle map to a triangle with angles specified in the claim. As the Schwarzian derivative determines a map up to Möbiustransformation, we are done. $\square$

Remark 2.10. The Theorem 2.9 allows for arbitarily large positive real numbers as angles. The existence of corresponding circular triangles is proven in [Kl].

2.4 Exploiting further symmetry: The case $s = t$.

In the case $s = t$, we are able to exploit further symmetry. This will allow us to determine the image of the upper half disk under period quotient maps. To understand this, we first recall the basic symmety properties of the modular λ-sphere:

For different values λ, λ', the corresponding tori can still be isomorphic, namely if and only if

$$\lambda' \in \left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, 1-\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}\right\}$$

This is reminiscent of the fact that λ captures the conformal structure of the torus as well as an ordered marking of the four 2-division points. Equivalently, while two lattice tori given by lattice constants τ and τ' are isomorphic if and only if

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSl(2,\mathbb{Z}),$$

they ony have the same modular invariant if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ where now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2 = \left\{A \in PSl(2,\mathbb{Z}) : A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2\right\}$$

The normal subgroup Γ_2 of $PSl(2,\mathbb{Z})$ has as quotient just the order six group given by the above explicit Möbiustransformations.

Furthermore, different tori T_λ and $T_{\lambda'}$ can be anti-isomorphic (i.e. there is an orientation reversing conformal diffeomorphism between them) if and only if $T_{\lambda'}$ is isomorphic to $T_{\bar\lambda}$. In terms of the lattice constant, the map $\tau \mapsto \bar\tau$ induces the map $\lambda \mapsto \bar\lambda$. This comes from the fact that the map $\tau \mapsto \lambda(\tau)$ can be defined as the cross ratio of the branch values of any degree two elliptic function on the torus.

So we see that fixed under the involution $\lambda \mapsto \bar\lambda$ are rectangular tori, characterized by the property that the fixed point set of an antiholomorphic involution has two components. Thus they are represented by a real modular invariant.

Similarly, fixed under the involutions $\lambda \mapsto 1/\bar\lambda$ are rhombic tori (see below).

As we have limited our discussions to the upper half plane, it seems natural to restrict the period quotient maps further to the upper half disk. This is possible in the symmetric case $s = t$, due to

Lemma 2.11.

$$\alpha_i(r,s;1/\lambda) = \lambda^{-r-2s+1/2}\alpha_{2-i}(r,s;\lambda)$$

so that in particular

$$\sigma(r,s;1/\lambda) = \frac{1}{\sigma(r,s;\lambda)}$$

Proof. For a path γ, we have with $u = \lambda x$

$$\begin{aligned}\int_\gamma \alpha(r,s;1/\lambda) &= \int_\gamma x^{r-1/2}(x-1)^{s-1/2}(x-\frac{1}{\lambda})^{s-1/2}dx \\ &= \lambda^{-r-2s+1/2}\int_{\lambda\cdot\gamma} u^{r-1/2}(u-\lambda)^{s-1/2}(u-1)^{s-1/2}du \\ &= \lambda^{-r-2s+1/2}\int_{\lambda\cdot\gamma} \alpha(r,s;\lambda)\end{aligned}$$

For γ a path encircling 0 and $1/\lambda$ (resp. 1), $\lambda\cdot\gamma$ is a path encircling 0 and 1 (resp. λ). The assertion follows. $\square$

This lemma will allow us to reduce the source domain of our period quotient maps from the upper half plane to the upper half disk, making the image domains smaller and more visible.

Corollary 2.12. *The period quotient map $\lambda \mapsto \sigma(r,s;\lambda)$ maps the upper half disk to a circular triangle with angles $\pi/2, |r+s|\pi, |s|\pi$ at the images of the points $-1, 0, 1$.*

2.5 Inductive period computation.

The next goal is to determine the position of the vertices of the image triangles, using our specific choice of cycles. For small values of r, s, this can be done easily by hand. For arbitary integers, we use the induction formulas from Proposition 2.16. To this end, we need to compare period functions for different sets of values r, s.

The periods $\alpha_i(r,s;\lambda)$ and the quotient maps $\sigma(r,s;\lambda)$ for different r, s are algebraically interrelated. We introduce the following notation:

Definition 2.13. Write

$$\alpha(r,s;\lambda) \approx \alpha(r',s';\lambda)$$

if the 1-forms have the same periods and

$$\alpha(r,s;\lambda) \sim \alpha(r',s';\lambda)$$

if both forms have the same period quotient.

The next identity is new and requires a general period relation argument:

Lemma 2.14. *Let f be a meromorphic function and β be a differential of the first or third kind (i.e. a meromorphic 1-form with at most simple poles) on a torus such that neither $f\beta$ nor $\frac{1}{f}\beta$ have residues. Then $f\beta \sim \frac{1}{f}\beta$.*

Proof. Let F be a primitive of $f\beta$ in a fundamental domain for the torus. F will be meromorphic because f has no residues. By the same argument as in the classical proof of the Legendre relation,

$$\begin{aligned}
(f\beta)_1\,(\beta/f)_2 - (f\beta)_2\,(\beta/f)_1 &= 2\pi i \sum_P \operatorname{res}_P \frac{F(z)}{f(z)}\beta \\
&= 2\pi i \sum_{f(P)\neq\infty} \operatorname{res}_P \frac{F(z)-F(P)}{f(z)}\beta \\
&\quad + 2\pi i \sum_{f(P)=\infty} \operatorname{res}_P \frac{F(z)}{f(z)}\beta \\
&= 0
\end{aligned}$$

The second identity holds because $\frac{1}{f}\beta$ has no residues, and all the residues in the last two sums vanish because at a regular point P of f, $(F(z)-F(P))/f(z)$ has at least a simple zero, cancelling a possible pole of β. This implies equality of the period quotients. $\square$

From this we obtain immediately

Corollary 2.15.

$$\alpha(r,s) \sim \alpha(-r,-s) \tag{2.13}$$

To derive the induction formulas, we express the periods of $\alpha(r,s)$ explicitly as a linear combination of the periods of$\alpha(r',s')$ and $\alpha(r'',s'')$ with coefficients rational in λ.

That this must be possible ist due to the fact that, on a torus, a suitable linear combination of any three differential forms without residues must be exact. In principle, the linear combination can be determined by a Legendre-type argument.

Proposition 2.16.

$$\alpha(r+2,s) \approx \frac{(\lambda+1)(r+s+1)}{r+2s+\frac{3}{2}}\alpha(r+1,s) - \lambda\frac{r+\frac{1}{2}}{r+2s+\frac{3}{2}}\alpha(r,s)$$
$$\alpha(r,s+1) \approx -\frac{(s+\frac{1}{2})(\lambda+1)}{r+2s+\frac{3}{2}}\alpha(r+1,s) + \lambda\frac{2s+1}{r+2s+\frac{3}{2}}\alpha(r,s)$$

Proof. Denote by d the exterior derivative on T_λ with respect to z and let $\wp'(z) = \frac{d}{dz}\wp(z)$. Then using the definition of $\alpha(r,s)$ in terms of $\wp$, differentiating with respect to z, extracting $\alpha(r-1,s-1)$ and applying the differential equations (2.2,2.3) for $\wp$ gives

$$\begin{aligned} d(\wp'\alpha(r,s)) &= \alpha(r-1,s-1)\Big(\wp'^2\big(r(\wp-1)(\wp-\lambda) \\ &\qquad + s\wp(\wp-\lambda) + s\wp(\wp-1)\big) + \wp''\wp(\wp-1)(\wp-\lambda)\Big)dz \\ &= \alpha(r,s)\Big(\big(r+2s+\frac{3}{2}\big)\wp^2 \\ &\qquad - (r+s+1)(\lambda+1)\wp + r\lambda + \frac{\lambda}{2}\Big)dz \end{aligned}$$

and proves the first formula. For the second, we just continue the calculation by rearranging the terms within the big brackets differently:

$$\begin{aligned} d(\wp'\alpha(r,s)) = \alpha(r,s)\Big(&\big(r+2s+\frac{3}{2}\big)(\wp-1)(\wp-\lambda) \\ &+ \big(s+s\lambda+\frac{\lambda+1}{2}\big)\wp - \big(2s+1\big)\lambda\Big) \quad \square \end{aligned}$$

2.6 Values of the period quotient map at the vertices.

Now we use the induction formulas from Proposition 2.16 to compute the values of the period quotient maps $\sigma(r,s;\lambda)$ at special points λ. We begin with $\lambda = 0, 1, \infty$, corresponding to degenerated tori.

Lemma 2.17.

$$\sigma(r,s;0) = 0$$
$$\sigma(r,s;1) = 1$$
$$\sigma(r,s;\infty) = \infty$$

Proof. The result is well known and easy to compute directly for $r = s = 0$. For the general case, we use the Proposition 2.16. The formulas simplify for $\lambda = 0$ to

$$\alpha(r+2,s;0) \approx \frac{(r+s+1)}{r+2s+\frac{3}{2}}\alpha(r+1,s;0)$$
$$\alpha(r,s+1;0) \approx -\frac{(s+\frac{1}{2})}{r+2s+\frac{3}{2}}\alpha(r+1,s;0)$$

so that the claim follows by induction. The case $\lambda = \infty$ is rduced to $\lambda = 0$ by Lemma 2.11. Finally, for $\lambda = 1$, recall that the two cycles γ_1 (resp. γ_2) defining the periods of our period quotient map are the cycles encircling 0 and 1 (resp. $1/\lambda$). so that the two period integrals converge to each other for $\lambda \to 1$. This is obvious in the case $s \geq 0$ and follows from Corollary 2.16 for $s < 0$. □

Lemma 2.18.
$$\sigma(r,s;-1) = (-1)^r i$$

Proof. We know this for $r = s = 0$ (it follows from the normalization of the periods). For $r = 1$ and $s = 0$, we know that the image of the upper half plane under the period quotient map is the circular triangle with vertices at $0, 1$ and ∞ and angles $\pi, 0$ and π. The double of this triangle is a model of the moduli space of tori using periods of ψ as coordinates. The square torus can be defined as the only torus being simultaneously rhombic and rectangular. Being rectangular (resp. rhombic) can be characterized as being fixed under the orientation reversing involution of the moduli space which fixes $0, 1$ and ∞ (resp. fixes 1 but exchanges 0 and ∞ for rhombic tori). The fixed point set of the 'rectangular' involution is the boundary of our $0, 1, \infty$-triangle, while the fixed point set of the 'rhombic' involution is the unit circle. The intersection of these two fixed point sets consists of the points $\pm i$. To determine the sign, we use the Legendre relation (2.6).

By the symmetry of the square torus and our sign conventions, $\omega_2(-1) = i\omega_1(-1)$ and $\psi_2(-1) = \pm i\psi_1(-1)$. From the Legendre relation (2.6) $\omega_1\psi_2 - \omega_2\psi_1 = 8\pi i$ it follows that in the latter equation we have in fact $\psi_2(-1) = -i\psi_1(-1)$ so that $\Psi(-1) = -i$.

Now we proceed by induction using the formulas from Proposition 2.16 which reduce at $\lambda = -1$ to

$$\alpha(r+2,s) \approx \frac{2r+1}{2(r+2s)+1}\alpha(r,s)$$
$$\alpha(r,s+1) \approx \frac{4s+2}{2r+4s+3}\alpha(r,s)$$

This implies the claim. □

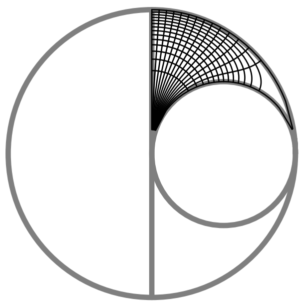
Put together, we obtain the main general statement of this section.

Theorem 2.19. *The period quotient map $\lambda \mapsto \sigma(r,s;\lambda)$ maps the upper half disk to a circular triangle with angles $\pi/2, |r+s|\pi, |s|\pi$ at the images of the points $-1, 0, 1$ which are the points $(-1)^r i, 0, 1$.*

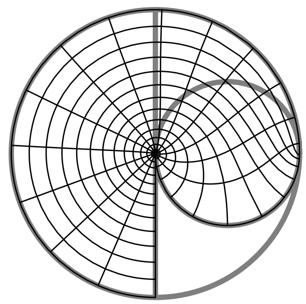
The following pictures illustrate this corollary. Observe that some domains are disjoint from the *complex conjugate* of other domains. This will become important in Section 3.

The first figure below shows one quarter of the classical fundamental domain for $\mathbb{H}/\Gamma(2)$, the moduli space of tori with marked 2-division points.

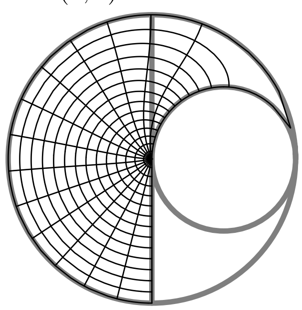
Fundamental Observation 2.20. In the first two rows, the complex conjugate of the left domain is disjoint of the right domain.



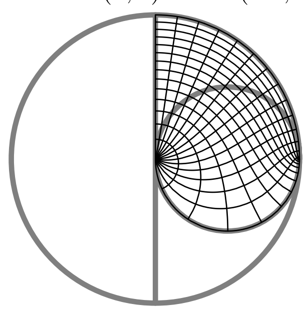
$\omega = \alpha(0, 0)$



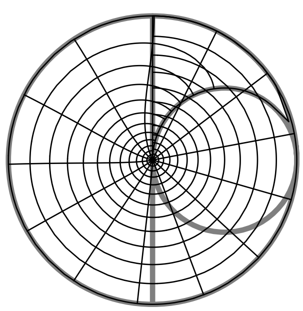
$\alpha(1, 1)$ and $\alpha(-3, 1)$



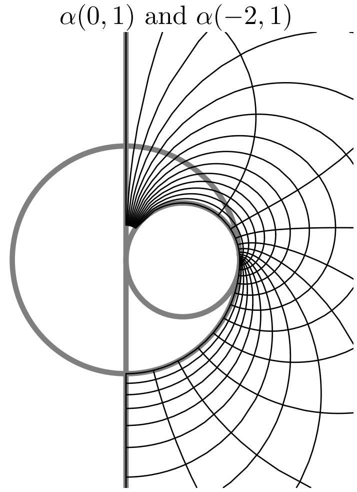
$\psi = \alpha(1, 0)$



$\alpha(0, 1)$ and $\alpha(-2, 1)$



$\alpha(2, 0)$



$\alpha(-1, 1)$

Remark 2.21. In the same way as four copies of the ω-domain tessellate the thrice punctured modular sphere to define a hyperbolic structure, the other domains can be used to construct different *projective* or *Möbius* structures on the same sphere.

2.7 A period quotient equation.

The following theorem is our main conclusion from the Observartion 2.20. We will apply it directly to minimal surface existence and uniqueness questions:

Theorem 2.22. *The only solution to the period-quotient equation*

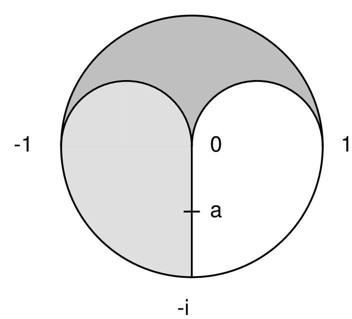
$$\overline{\sigma(1,0;\lambda)} = \sigma(0,1;\lambda) \tag{2.14}$$

is $\lambda = -1$.

Proof. For λ in the open upper half disk, the image domains of $\sigma(1,0,\cdot)$ and $\overline{\sigma(0,1,\cdot)}$ are clearly disjoint. This still holds for $\lambda \in (-\infty,-1)$. If we can show that this holds also for $\lambda \in (-1,0]$ and $\lambda \in [0,1]$, we are done, because both maps are continued by reflection. So using the geometric properties of the period quotient maps, we have reduced a 2-dimensional problem to a 1-dimensional one, which is also tractable by direct arguments because the moduli λ on the boundary represent rectangular tori. To see the two remaining assertions, we assume the contrary. Suppose first that there is a point $\lambda_0 \in [-1,0]$ so that $a := \sigma(1,0,\lambda_0) = \overline{\sigma(0,1;\lambda_0)}$. This means that the two domains with vertices $0,-1,-i$ (lightly shaded in the figure below) and $0,1,-i$ (consisting below of the light and dark shaded regions) are conformally equivalent by a map fixing $0,-i$ and a and mapping -1 to 1. Now consider the family of curves Γ_1 connecting the segment $[0,a]$ with the segment $[-1,-i]$ on the unit circle and the family Γ_2 connecting $[0,a]$ with $[-1,1]$ on the unit circle. By [Ahl] Theorem 4b, we have for the extremal lengths

$$\frac{1}{\mathrm{Ext}(\Gamma_1\cup\Gamma_2)} \geq \frac{1}{\mathrm{Ext}(\Gamma_1)} + \frac{1}{\mathrm{Ext}(\Gamma_2)}$$

On the other hand, by assumption $\mathrm{Ext}(\Gamma_1\cup\Gamma_2) = \mathrm{Ext}(\Gamma_1)$ and $\mathrm{Ext}(\Gamma_2) > 0$, which is a contradiction.



Finally, let $\lambda \in (0,1]$. We will explicitly estimate the period integrals for $\alpha(1,0;\lambda)$ and $\alpha(0,1;\lambda)$. For $\rho = \frac{1-\lambda}{3}$ we have by comparing the integrands

$$(\rho + \frac{\lambda+1}{3})\int_0^\lambda \sqrt{\frac{x}{(1-x)(\lambda-x)}}dx < \frac{2\lambda}{3}\int_0^\lambda \frac{dx}{\sqrt{x(1-x)(\lambda-x)}}$$
$$(\rho + \frac{\lambda+1}{3})\int_\lambda^1 \sqrt{\frac{x}{(1-x)(x-\lambda)}}dx > \frac{2\lambda}{3}\int_\lambda^1 \frac{dx}{\sqrt{x(1-x)(x-\lambda)}}dx$$

By the second formula of Proposition 2.16 for $r = s = 0$, we have for any cycle γ

$$\int_\gamma \alpha(0,1;\lambda) = -\frac{\lambda+1}{3}\int_\gamma \psi + \frac{2\lambda}{3}\int_\gamma \omega$$

This implies

$$\rho\int_0^\lambda \sqrt{\frac{x}{(1-x)(\lambda-x)}}dx < \int_0^\lambda \sqrt{\frac{(1-x)(\lambda-x)}{x}}dx$$
$$\rho\int_\lambda^1 \sqrt{\frac{x}{(1-x)(x-\lambda)}}dx > \int_\lambda^1 \sqrt{\frac{(1-x)(x-\lambda)}{x}}dx$$

which shows that the period quotients of $\alpha(1,0;\lambda)$ and $\alpha(0,1;\lambda)$ cannot be complex conjugate for $\lambda \in (0,1]$. □

3. Applications to minimal surfaces

In this section, we discuss applications of the above theorems to minimal surfaces defined on tori.

3.1 Minimal Surface Background.
The theory of complete minimal surfaces of finite total curvature is related to complex analysis like no other subject in differential geometry: Any such minimal surface will be given by a triple $\omega_1, \omega_2, \omega_3$ of meromorphic 1-forms defined on some Riemann surface such that

(3.1) $\omega_1^2 + \omega_2^2 + \omega_3^2 = 0;$

(3.2) all periods of the ω_i are purely imaginary.

Then the surface can be parameterized by

$$z \mapsto \mathrm{Re}\int^z (\omega_1, \omega_2, \omega_2) \tag{3.3}$$

Usually, the first requirement (which ensures the conformality of the map (3.3)) is guaranteed by introducing the Weierstraß representation formulas

$$\omega_1 = \left(\frac{1}{G} - G\right)dh, \quad \omega_2 = i\left(G + \frac{1}{G}\right)dh, \quad \omega_3 = 2dh$$

with a meromorphic function G (the conformal Gauß map) and meromorphic 1-form dh (the height differential). The second condition on the periods can then be replaced by the requirements that the periods of Gdh and $\frac{1}{G}dh$ are complex conjugate and the periods of dh are purely imaginary.

The classical examples of complete minimal surfaces of finite total curvature are all defined on the sphere and their meromorphic Weierstraß data can be written down in terms of rational functions. All periods are computable as residues, and existence or uniqueness questions become essentially purely algebraic problems.

The situation changes when one considers the higher genus examples found in the last three decades. The first immersed minimal torus was the Chen-Gackstatter surface, and the first *embedded* torus was the Costa surface. The Weierstraß data consist in both cases of elliptic functions on a square torus. The existence of these surfaces can be deduced without heavy computations only because the symmetries of the Weierstraß data guarantee that the period conditions are satisfied automatically. However, it is *much* harder to show that these surfaces are the *unique* surfaces with their given geometric properties. For still higher genus, already the existence problem becomes very hard, and presently there is not much known about uniqueness questions.

The known general uniqueness proofs for minimal surfaces on tori are due to Costa ([Cos2]), Bloss ([Bl]) and Lopéz ([Lo]). They heavily make use of special estimates of elliptic functions and theta functions and don't shed much light on the question why such theorems should hold. It is the major intent of this paper to give an answer to this question. Especially, using our methods, it is now possible to give a complete conceptual uniqueness proof for the chen-Gackstatter surface. To this end, we have developed in the previous section a geometric theory for *period quotient maps* which map the modulus of a torus to the quotient of the periods of certain meromorphic forms on that torus. This mapping behavior encapsulates the transcendental information of the period integrals relevant for minimal surfaces. In many cases these maps were geometrically easy to describe and allowed to obtain non-trivial conclusions about the solution of period problems.

For background on complete minimal surfaces of finite total curvature, we refer the reader to [HK] and [Ka2].

We will consider the Weierstraß representation of a surface as

$$z \mapsto \mathrm{Re} \int^z \left(\frac{1}{G} - G, i(G + \frac{1}{G}), 2 \right) dh$$

where G is a meromorphic function on a compact Riemann surface and dh a meromorphic 1-form.

Denote by $\kappa = \int |K|$ the total absolute curvature of the surface.

We are interested in complete immersed surfaces of finite total curvature. This imposes three restrictions on the Weierstraß data G and dh:

Conditions on Weierstrass Data.

(1) *The Riemannian metric, given by*

$$ds^2 = \left(|G|^2 + \frac{1}{|G|^2} \right) |dh|^2$$

must be complete and non-degenerate.

(2) *The periods of Gdh and $\frac{1}{G}dh$ must be complex conjugate.*

(3) *The periods of dh must be purely imaginary.*

The second condition can often (in our cases always) be replaced by the condition

(2*) *The period quotients of Gdh and $\frac{1}{G}dh$ must be complex conjugate.*

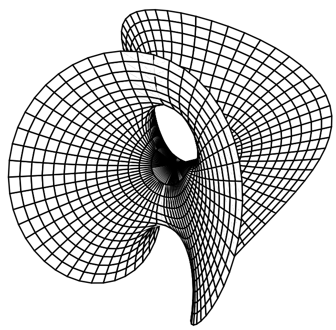
This being satisfied one can often rescale the Gauß map G to ρG using the so-called Lopez-Ros parameter ρ so that the periods of Gdh and $\frac{1}{G}dh$ become conjugate (and so satisfy (2*)).

One possible classification approach for minimal surfaces deduces from assumptions on the geometry of the surface the possible divisors of the Weierstraß data, using the completeness condition. In a good situation, one is then left with the moduli of the underlying Riemann surface as free parameters to satisfy the period conditions.

This will be in many cases a high-dimensional nonlinear problem. Our discussion focuses on the complex 1-dimensional case, namely the moduli space of tori.

3.2. On the uniqueness of the Chen-Gackstatter surface.

The Chen-Gackstatter surface ([CG]) was the first immersed minimal torus, it is known to be the unique complete minimal surface of genus 1 with minimal total absolute curvature 8π. The surface has one Enneper type end (hence it is not embedded). The surface lives on the square torus $T_{-1} : y^2 = x(x^2-1)$ with Weierstraß data $G = x/y$ and $dh = \alpha(1,1;-1)$.



After Osserman had proven his famous theorem characterizing complete minimal surfaces of finite total curvature, it became a natural question to classify at least all such surfaces with *small* total absolute curvature. This number is the same as $4\pi \deg G$, and the cases $\kappa = 0$ (the euclidean plane) and $\kappa = 4\pi$ (the catenoid and Enneper's surface) were already treated by Osserman ([Oss]). While there are many *spheres* with $\kappa = 8\pi$, the Chen-Gackstatter surface provides an example of a torus with this total curvature. As proven later (see [Bl, Lo]), this is the only example with these properties. The proof requires two case distinctions, and each case is long and tedious without providing much insight *why* such a uniqueness result should hold. Using the period quotient technique, we can treat one of the cases in a surprisingly geometric manner. The other case can be treated by a beautiful and very different geometric argument due to Kusner, see the proof of (their) case 1 in theorem 5.19 in the survey article [LM]. Both these arguments together yield a complete conceptual proof of the unicity of the Chen-Gackstatter surface.

We will now prove:

Theorem 3.1. *Let X be a complete minimal surface of total absolute curvature $\kappa = 8\pi$ defined on a torus with one end where the Gauß map has order* 1 *(first case). Then X is the Chen-Gackstatter surface.*

Proof. Denote by E the puncture of the torus where X has its end. By rotating the surface if necessary, we can assume that $G(E) = \infty$. Because of $\deg G = 2$, either $G(E) = \infty^1$ or $= \infty^2$ (untreated). Our assumption allows us to omit a discussion of the second case. Hence there is another point P with $G(P) = \infty^1$ and two points V_i with $G(V_i) = 0^1$. We allow for the moment that possibly $V = V_1 = V_2$ and then

$G(V) = 0^2$. The Riemannian metric $ds = 2(|G| + 1/|G|)|dh|$ being complete forces dh to have simple zeroes at C_i and P, and the metric being non-degenerate implies that there are no other zeroes elsewhere. The degree of the divisor of dh being zero, we conclude that

$$(dh) = -3E + C_1 + C_2 + P \quad \text{and} \quad (G) = -E + C_1 + C_2 - P$$

and therefore $(\frac{1}{G}dh) = -2E + 2N$. Now we normalize the torus so that $E = 0$ and $N = (\tau + 1)/2$. Hence for some $a \neq 0$, we can write $\frac{1}{G}dh = a\wp dz$. Our next goal will be to show that also the C_i are 2-division points: We can write

$$\begin{aligned} dh &= bd\wp + c\wp dz \\ &\approx c\wp dz \end{aligned}$$

for some $b \neq 0$, and we have to show that $c = 0$. We obtain

$$\begin{aligned} G &= \frac{b\wp' + c\wp}{a\wp} \\ Gdh &= \frac{(b\wp' + c\wp)^2}{a\wp} dz \\ &= \left(\frac{b^2\wp(\wp - 1)(\wp - \lambda)}{a\wp} + \frac{2bc\wp\wp'}{a\wp} + \frac{c^2\wp}{a} \right) dz \\ &= \frac{b^2}{a}\alpha(0, 1; \lambda) + \frac{2bc}{a}\wp' dz + \frac{c^2}{a}\wp dz \\ &\approx \frac{b^2}{a}\left(-\frac{1}{3}(\lambda + 1)\wp dz + \frac{2}{3}\lambda dz \right) + \frac{c^2}{a}\wp dz \end{aligned}$$

where we have used the second formula of Proposition 2.16 in the case $r = s = 0$ (see also Corollary 4.1) in the last step. The periods of dh being purely imaginary implies that its period quotient must be real. This being the same as the period quotient of Gdh, we conclude that also the period quotient of $\frac{1}{G}dh$ is real and coincides with that of Gdh. Especially, using the Legendre relation (2.6), we get

$$\begin{aligned} 0 &= Gdh_1 \frac{1}{G}dh_2 - Gdh_2 \frac{1}{G}dh_1 \\ &= \frac{2b^2c}{3a}\lambda(\omega_1\psi_2 - \omega_2\psi_1) \\ &= 8\pi i \frac{2b^2c}{3a}\lambda \end{aligned}$$

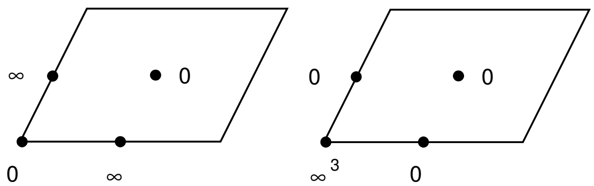
which implies $c = 0$. Hence we have reduced the divisors of the candidate Weierstraß data to the situation treated by the next lemma:

Lemma 3.2. *Suppose that on the torus T_λ there are Weierstraß data given by*

$$\begin{aligned} Gdh &= \rho\alpha(1, 0; \lambda) \\ \frac{1}{G}dh &= \frac{1}{\rho}\alpha(0, 1; \lambda) \\ dh &= \alpha(1, 1; \lambda) \\ G &= x/y = \wp/\wp' \end{aligned}$$

Then the conditions (1) and (3) are always satisfied while condition (2) is satisfied for suitable ρ if and only if $\lambda = -1$.

Proof. The divisors of G, $\frac{1}{G}$ and dh cancel in the expression of ds^2 at the non-zero 2-division points of T_λ and give a fourth pole at 0 of ds^2 representing the end:

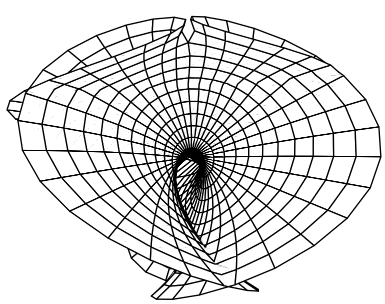


Divisors for G and dh

The 1-form dh is exact, and Theorem 2.22 gives the claim about the conjugate period quotients. □

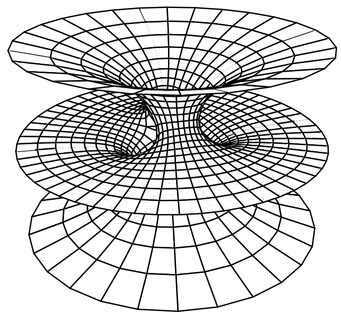
We finish this subsection with a brief discussion of a

Related Example 3.3. As already noticed, also for the 1-forms $Gdh = \rho\omega$ and $\frac{1}{G}dh = \frac{1}{\rho}\alpha(1,1)$ the period quotients are complex conjugate if and only if $\lambda = -1$. All the other period conditions are always satisfied (dh becomes the exact form $\wp'(z)dz$, and $G = 1/\wp'$), and the reader might wonder to which minimal surface these Weierstraß data correspond to. This surface apparently hasn't been noticed before, it is an immersed torus with one Enneper end of winding number 5 and total absolute curvature 12π:



3.3. On the uniqueness of the Costa surface.

For higher total curvature, it becomes more and more easy to construct minimal surfaces, unless one restricts oneself to the most interesting class of *embedded* tori.



Here, the Costa surface ([Cos1]) was the first embedded minimal torus. It lives on the square torus, has total absolute curvature $\kappa = 12\pi$, two catenoid ends and one planar end. Later (see [HM]) the planar end was shown to be deformable into a catenoid end, giving rise to a 3-ended embedded minimal surface for each rectangular torus. Then, Costa ([Cos2]) has shown that these are in fact *all* 3-ended (alomost) embedded minimal tori. The classification requires this time three case distinctions corresponding to the order of the Gauß map at the middle end being $1, 2$ or 3, of which we can handle only one case. More precisely, we will prove:

Theorem 3.4. *Let X be a complete minimal torus of finite total curvature with three parallel embedded ends. Assume that the Gauß map has order* 3 *in the middle (planar) end. Then X is the Costa surface.*

Proof. We normalize the Gauß map so that at the planar end P it has the value $G(P) = \infty^3$, and at the catenoid ends C_i we have $G(C_i) \in \{0^1, \infty^1\}$. The ends being embedded, they have logarithmic growth rate and hence dh must have a single zero at P and single order poles at C_i. Because there are no other ends, dh has no further poles, hence must have another simple zero at some point V where the normal is also vertical. To compensate the triple order pole at P, G must have simple zeroes at C_i and V. We conclude

$$\begin{aligned}
(G) &= -3P + C_1 + C_2 + V \\
(dh) &= P - C_1 - C_2 + V \\
(Gdh) &= -2P + 2V \\
\left(\frac{1}{G}dh\right) &= 4P - 2C_1 - 2C_2
\end{aligned}$$

We normalize the torus so that $P = 0$ and $V = (\tau + 1)/2$. Again our next goal will be to show that also the C_i are 2-division points. Note that Gdh has no residues, and because *all* the periods of Gdh are conjugate to those of $\frac{1}{G}dh$, also $\frac{1}{G}dh$ has no residues. Hence we can write

$$f(z) := \frac{1}{G}\frac{dh}{dz} = a_1\wp(z + t_1) + a_2\wp(z + t_2)$$

with $a_i \neq 0$ and $t_i \in T_\lambda$. This function f has by assumption a fourth order zero at $z = P = 0$. Using the differential equation for $\wp(z)$ and differentiating further (see

Lemma 2.1), we get

$$\wp^{(3)}(z) = \wp'(z)(2\wp(z) - (\lambda+1))$$

and hence at $z = 0$

$$\begin{aligned}
0 &= f^{(3)}(0)\\
&= a_1\wp^{(3)}(t_1) + a_2\wp^{(3)}(t_2)\\
&= a_1\wp'(t_1)(3\wp(t_1) - (\lambda+1)) + a_2\wp'(t_2)(3\wp(t_2) - (\lambda+1))\\
&= 3a_1\wp'(t_1)\wp(t_1) + 3a_2\wp'(t_2)\wp(t_2) - (\lambda+1)a_1\wp'(t_1) - (\lambda+1)a_2\wp'(t_2)\\
&= 3a_1\wp'(t_1)(\wp(t_1) - \wp(t_2))
\end{aligned}$$

If $\wp(t_1) = \wp(t_2) = 0$, we have $t_1 = t_2 = 0$ contradicting the requirement on $f(z)$ having a fourth order zero. If $\wp(t_1) = \wp(t_2) \neq 0$, we have $t_2 = -t_1 \neq 0$ and $a_2 = -a_1$ to make $f(z)$ vanish at $z = 0$. But then $f(z) = a_1(\wp(z+t_1) - \wp(z-t_1))$ which does not vanish in general to the fourth order at $z = 0$. Hence $\wp'(t_1) = 0$ and by the symmetry of the construction also $\wp'(t_2) = 0$ meaning that the t_i are 2-division points as claimed. These considerations have reduced the assertion to the following

Lemma 3.5. *Suppose on the torus T_λ Weierstraß data are given by*

$$\begin{aligned}
Gdh &= \rho\alpha(1,0;\lambda)\\
\frac{1}{G}dh &= \frac{1}{\rho}\alpha(0,-1;\lambda)\\
dh &= \frac{dx}{(x-1)(x-\lambda)}\\
G &= y
\end{aligned}$$

Then the condition (1) is always satisfied, condition (3) is valid if and only if λ is real, and condition (2) is satisfied for suitable ρ if and only if $\lambda = -1$

Proof. The divisors of G and dh cancel in the expression of ds^2 at $(\tau+1)/2$ and give a double order pole for ds at the other three 2-division points, providing the necessary behavior at the ends The 1-form dh is expressed in terms of x, hence defined already on the x-sphere, where its periods can be evaluated as residues. These residues are real if and only if λ is real. By Corollary 2.15, $\frac{1}{\rho}\alpha(0,-1;\lambda) \sim \frac{1}{\rho}\alpha(0,1;\lambda)$, and Theorem 2.22 applies (again!) to show the last assertion. □

3.4 On the existence of the singly-periodic helicoid with handle.

The singly periodic helcioid with handle is an embedded minimal surface which is invariant under a translation so that the quotient surface has genus one and two helicoidal ends. It was discovered by Hoffman, Karcher and Wei ([HKW1,HKW3]), and it is closely related to the Genus One Helicoid ([HKW1,HKW2]). The solution of the horizontal period problem for this surface requires to find a rhombic torus such that the meromorphic form $\wp(z)dz$ has real period quotient. In the original paper [HKW1], this fact was barred behind tedious computations, and it was not apparent that this conditions determines the torus uniquely:

Proposition 3.6. *There exists a rhombic torus T_λ, unique up to isomorphism, so that the meromorphic form $\wp(z)dz$ has real period quotient.*

Proof. In the period quotient picture for ψ, the tori with real period quotient lie on the real axes. The rhombic tori lie on the unit circle, and the only nondegenerate intersection point is -1.

This observation has simplified the existence proof of the singly-periodic helicoid with handle as well as of oter related surfaces, which can be used to prove the embeddednes of the Genus One Helicoid, see [W8, WHW].

4. Families of period quotient maps

In this section, we will generalize the period quotient mapping Theorem 2.19 to 1-forms which are linear combinations of the type

$$\mu P(\lambda)\psi + \omega$$

for some *fixed* rational function $P(\lambda)$. Again it is possible to compute the Schwarzian derivative of the corresponding period quotient map explicitly, however the formulas become very complicated and it is not easy to see algebraically which families yield image domains which are simple enough for applications. Our choices are motivated by the following corollary to Proposition 2.16:

Recall that we denote $\omega = \alpha(0,0) = dz$ and $\psi = \alpha(1,0) = \wp(z)dz$.

Corollary 4.1.

$$\begin{aligned}
\alpha(0,1) &\approx -\frac{1}{3}(\lambda+1)\psi + \frac{2}{3}\lambda\omega \\
\alpha(2,0) &\approx \frac{2}{3}(\lambda+1)\psi - \frac{1}{3}\lambda\omega \\
\alpha(-2,1) &\approx -\frac{\lambda+1}{3\lambda}\psi + \frac{2}{3}\omega \\
\alpha(1,1) &\approx -\frac{2}{15}(\lambda^2-\lambda+1)\psi + \frac{1}{15}\lambda(\lambda+1)\omega \\
\alpha(-3,1) &\approx -\frac{2}{15\lambda^2}(\lambda^2-\lambda+1)\psi + \frac{1}{15\lambda}(\lambda+1)\omega \\
\alpha(-1,1) &\approx 2\psi - (\lambda+1)\omega
\end{aligned}$$

Remark. As a surprising byproduct, we obtain that the period relations for some of the $\alpha(r,s)$ belong to certain linear 1-parameter families of 1-forms of the type

$$\mu P(\lambda)\psi + \omega$$

with the *same* rational function $P(\lambda)$. This suggests to investigate the mapping behaviour of all maps in these linear families, which is what we will do now.

In particular, we find that the period quotients of the 1-forms $\omega = \alpha(0,0)$, $\psi = \alpha(1,0)$, $\alpha(1,1)$ and $\alpha(2,0)$ belong all to the same family

$$\beta^\mu = \mu \cdot (\lambda+1)\psi + \lambda\omega$$

for the values $\mu = 0, \infty, -1/2$ and $\mu = -2$, respectively. We will also discuss the family

$$\gamma^\mu = \mu\psi + (\lambda+1)\omega$$

which passes through $\omega = \alpha(0,0), \psi = \alpha(1,0)$ and $\alpha(-1,1)$ for $\mu = 0, \infty$ and $\mu = -2$, respectively.

We begin by deducing the symmetry properties of these new period quotient maps.

Denote by

$$B^\mu(\lambda) = \beta_2^\mu(\lambda)/\beta_1^\mu(\lambda)$$
$$\Gamma^\mu(\lambda) = \gamma_2^\mu(\lambda)/\gamma_1^\mu(\lambda)$$

the period quotient maps corresponding to the 1-forms β^μ and γ^μ.

Lemma 4.2.

$$B^\mu\left(\frac{1}{\lambda}\right) = 1/B^\mu(\lambda)$$
$$\Gamma^\mu\left(\frac{1}{\lambda}\right) = 1/\Gamma^\mu(\lambda)$$

Proof. By Lemma 2.11,

$$\omega_i\left(\frac{1}{\lambda}\right) = \lambda^{1/2}\omega_{2-i}(\lambda)$$
$$\psi_i\left(\frac{1}{\lambda}\right) = \lambda^{-1/2}\psi_{2-i}(\lambda)$$

so that

$$\begin{aligned}\beta_i^\mu\left(\frac{1}{\lambda}\right) &= \mu\left(\frac{1}{\lambda}+1\right)\psi_i\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda}\omega_i\left(\frac{1}{\lambda}\right)\\ &= \mu\frac{\lambda+1}{\lambda}\lambda^{-1/2}\psi_{2-i}(\lambda) + \frac{1}{\lambda}\lambda^{1/2}\omega_{2-i}(\lambda)\\ &= \lambda^{-3/2}\beta_{2-i}^\mu(\lambda)\end{aligned}$$

which implies the first claim. The second is proven similarly. □

This symmetry lemma allows us to deduce where the boundary segments of the upper half disk are mapped to. The global mapping behavior will follow again from a Schwarz-Christoffel argument. To use it, we first compute the Schwarzian derivative of $\Gamma^\mu(\lambda)$ and $B^\mu(\lambda)$. For the behavior of the period quotient map, the angle is determined by the residue of the Schwarzian derivative (as a quadratic differential).

The Schwarzian derivative can be computed (see Lemma 4.4 below) for quotients of functions which satisfy a second order linear differential equation. The problem is to find the coefficients of this differential equation explicitly. This can be automatized, and we give the result with all necessary information to check the claims by hand:

Lemma 4.3. *The periods of* β^μ *satisfy the second order differential equation*

$$A\beta + B\beta' + C\beta'' = 0$$

with

$$\begin{aligned} A &= -3\mu(\mu+2)\lambda^2 + (14\mu^2 + 14\mu - 1)\lambda + (\mu+2)^2 \\ B &= 4\lambda\left(\mu(\mu+2)\lambda^2 - 2\mu(\mu+2)\lambda - 3\mu^2 - 2\mu - 1\right) \\ C &= -4\lambda(\lambda-1)\left(\mu(\mu+2)\lambda^2 + (2\mu^2+1)\lambda + \mu(\mu+2)\right) \end{aligned}$$

and the periods of γ^μ *satisfy*

$$A\gamma + B\gamma' + C\gamma'' = 0$$

with

$$\begin{aligned} A &= \lambda^2 - (\mu^2 + 6\mu + 10)\lambda + 2\mu + 5 \\ B &= 4\left((\mu+1)\lambda + 1\right)\left((\mu+3)\lambda - 1\right) \\ C &= 4\lambda(\lambda-1)\left(\lambda^2 + (\mu^2 + 4\mu + 2)\lambda + 1\right) \end{aligned}$$

Proof. We first prove the statement for β^μ: Let

$$\beta^\mu = f dz = z^{-1/2}(z-1)^{-1/2}(z-\lambda)^{-1/2}(\mu z(\lambda+1) + \lambda)dz$$

and

$$\begin{aligned} Q &= 2\lambda\left(\mu(11\mu+4)\lambda^2 - (6\mu^2 - 2\mu - 1)\lambda - \mu(\mu+2)\right) \\ h_1 &= \mu\frac{\mu(\mu+2)\lambda^3 + (3\mu^2 - 8\mu - 1)\lambda^2 + (3\mu^2 + 10\mu - 1)\lambda + \mu^2 + 4\mu + 4}{\lambda\left(\mu(11\mu+4)\lambda^2 - (6\mu^2 - 2\mu - 1)\lambda - \mu(\mu+2)\right)} \end{aligned}$$

Then it is straightforward but tedious to check that

$$Af + Bf' + Cf'' = Q\frac{d}{dz}\left(z^{1/2}(z-1)^{1/2}(z-\lambda)^{1/2}(1 + h_1 z)\right)$$

which implies the claim by integration because Q is independent of z.

Similarly for γ^μ: Let

$$\gamma^\mu = f dz = z^{-1/2}(z-1)^{-1/2}(z-\lambda)^{-1/2}(\mu z + \lambda + 1)dz$$

and

$$\begin{aligned} Q &= -2\left((2\mu+1)\lambda^3 - (\mu^2 + 4\mu - 3)\lambda^2 + (3\mu^2 + 10\mu + 3)\lambda + 1\right) \\ h_1 &= -\frac{\mu\left(\lambda^2 - (\mu^2 + 6\mu + 10)\lambda + 2\mu + 5\right)}{(2\mu+1)\lambda^3 - (\mu^2 + 4\mu - 3)\lambda^2 + (3\mu^2 + 10\mu + 3)\lambda + 1} \end{aligned}$$

Then it is similarly tedious to check that

$$Af + Bf' + Cf'' = Q\frac{d}{dz}\left(z^{1/2}(z-1)^{1/2}(z-\lambda)^{1/2}(1 + h_1 z)\right)$$

which implies again the claim by integration. □

Using the coefficients of a differential equation, the Schwarzian derivative of quotients of independent solutions can be obtained by Lemma 2.6. As a corollary, we get

Corollary 4.4.

$$\begin{aligned}\mathcal{S}_\lambda B^\mu(\lambda) = &\frac{\mu+2}{2\mu\lambda} + \frac{1}{2(\lambda-1)^2} - \frac{1}{2(\lambda-1)}\\ &+ \frac{3(2\mu+1)^2(4\mu-1)}{2\left(\mu(\mu+2)\lambda^2+(2\mu^2+1)\lambda+\mu(\mu+2)\right)^2}\\ &- \frac{\mu(\mu+2)\lambda+2\mu^2+1}{\mu\left(\mu(\mu+2)\lambda^2+(2\mu^2+1)\lambda+\mu(\mu+2)\right)}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_\lambda \Gamma^\mu(\lambda) = &\frac{1}{2\lambda^2} + \frac{2\mu^2+6\mu+1}{2\lambda} + \frac{1}{2(\lambda-1)^2} - \frac{1}{2(\lambda-1)}\\ &- \frac{3\mu(\mu+2)^2(\mu+4)}{2\left(\lambda^2+(\mu^2+4\mu+2)\lambda+1\right)^2} - \frac{\mu(\mu+3)(\lambda+\mu^2+4\mu+2)}{\lambda^2+(\mu^2+4\mu+2)\lambda+1}\end{aligned}$$

Proof. By a lengthy computation, which is omitted. □

The purpose of this computation was to obtain the *residues* of the Schwarzian derivatives, because these are responsible for the angles at the vertices of the circular image domains, see lemma 4.8 below. Some of the branch points are only solutions of quadratic equations, but the residues there can be obtained explicitly by

Lemma 4.5. *Let $az^2+bz+c = a(z-z_0)(z-z_1)$. Then*

$$\operatorname{res}_{z=z_0} \frac{dz}{(az^2+bz+c)^2} = \frac{1}{b^2-4ac}$$

Proof. Recall that a residue of a quadratic differential is its coefficient of z^{-2} in a local coordinate z.

$$\begin{aligned}\operatorname{res}_{z=z_0} \frac{dz}{(az^2+bz+c)^2} &= \operatorname{res}_{z=z_0} \frac{dz}{(a(z-z_0)(z-z_1))^2}\\ &= \frac{1}{(a(z_0-z_1))^2}\\ &= \frac{1}{b^2-4ac}\end{aligned}$$

□

Corollary 4.6. *Consider the period quotient family B^μ. For generic μ ,*

$$\operatorname{res}_{\lambda_{0,1}} \mathcal{S}_\lambda B^\mu(\lambda) = -\frac{3}{2}\ .$$

Observe that the expression for $\mathcal{S}_\lambda B^\mu(\lambda)$ degenerates for $\mu = -1/2, 1/4, 0, \infty$. In these cases, we compute

$$\begin{aligned}\mathcal{S}_\lambda B^{-1/2}(\lambda) &= -\frac{3}{2\lambda} - \frac{3}{2(\lambda-1)^2} + \frac{3}{2(\lambda-1)}\\ \mathcal{S}_\lambda B^{1/4}(\lambda) &= \frac{9}{2\lambda} + \frac{1}{2(\lambda-1)^2} - \frac{1}{2(\lambda-1)} - \frac{4}{\lambda+1)^2} - \frac{4}{\lambda+1}\end{aligned}$$

Hence for all μ,

$$\operatorname{res}_\infty \mathcal{S}_\lambda B^\mu(\lambda) = 0\ .$$

Corollary 4.7. *Consider the period quotient family* Γ^μ. *For generic* μ ,

$$\mathrm{res}_{\lambda_{0,1}}\{\Gamma^\mu(\lambda),\lambda\} = -\frac{3}{2}\ .$$

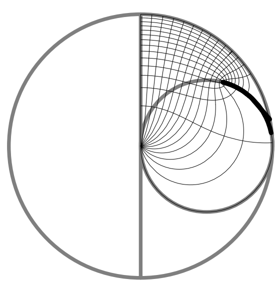
Observe that the expression for $\mathcal{S}_\lambda\Gamma^\mu(\lambda)$ *degenerates for* $\mu = 0, -2, -4, \infty$. *In these cases, we compute*

$$\mathcal{S}_\lambda\Gamma^{-2}(\lambda) = \frac{1}{2\lambda^2} - \frac{3}{2\lambda} - \frac{3}{2(\lambda-1)^2} + \frac{3}{2(\lambda-1)}$$
$$\mathcal{S}_\lambda\Gamma^{-4}(\lambda) = \frac{1}{2\lambda^2} + \frac{9}{2\lambda} + \frac{1}{2(\lambda-1)^2} - \frac{1}{2(\lambda-1)} - \frac{4}{(\lambda+1)^2} - \frac{4}{\lambda+1}$$

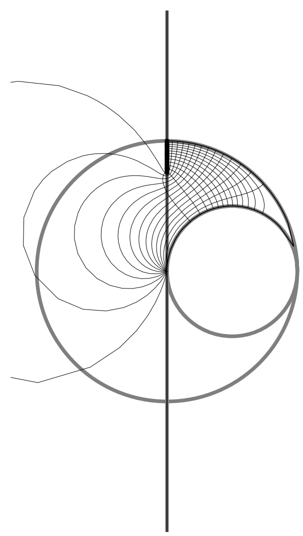
Hence for all μ,

$$\mathrm{res}_\infty\, \mathcal{S}_\lambda\Gamma^\mu(\lambda) = \frac{1}{2}\ .$$

Below are some pictures illustrating this mapping behavior. It is maybe helpful to think of the image domains as flooded areas and the segments containing the 2π corners as gates opening and closing the subdomains as μ varies in $\hat{\mathbb{R}}$, causing drastic changes whenever a gate is newly opened.



B^μ for $\mu = -0.25$



B^μ for $\mu = 0.05$

Knowing the residues of a Schwarz-Christoffel map allows to deduce its mapping behavior near the singular points:

Lemma 4.8. *Suppose that* f *is a meromorphic map,* $\mathcal{S}_z f(z)$ *is real and that* $\mathrm{res}_{z=z_0}\mathcal{S}_z f(z) = \frac{1}{2}(1-\rho^2)$. *Then the segment* $(z_0-\epsilon, z_0+\epsilon)$ *on the real axes is mapped to circular (or straight) segments meeting at* $f(z_0)$ *with angle* $\rho\pi$.

Proof. All local solutions of the ordinary differential equation $\mathcal{S}_z g(z) = \mathcal{S}_z f(z)$ near z_0 are obtained by postcomposing g with Möbiustransformations. Since the Schwarzian derivative $\mathcal{S}_z f(z)$ is real, solutions in the upper half plane can be continued to the lower half plane by reflections at circles. This implies that real axes segments are mapped to circular or straight segments. The angle at which the segments intersect at $f(z_0)$ can be computed by postcomposing f with an appropriate power of $z-f(z_0)$ and using the composition rule for the Schwarzian derivative. The claim follows from the uniqueness theorem for ordinary differential equations. □

Using this, we can now describe the mapping given by the period quotient maps B^μ and Γ^μ:

Theorem 4.9. *The period quotient map B^μ maps the upper half disk to a circular quadrilateral so that (angles are denoted in brackets)*

$$0 \mapsto 0\ (\pi), \quad 1 \to 1\ (0), \quad -1 \mapsto i\ (\pi/2).$$

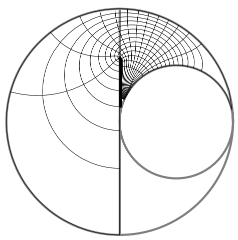
For $\mu \in (-\frac{1}{2}, \frac{1}{4})$, the branch points z_i are real, and one of them lies in the interval $[-1,1]$. The image of this branch point becomes a 2π vertex. For $\mu < 0$, this branch point is in $(0,1)$ and hence mapped to the smaller circle. For $\mu > 0$, it lies in $(-1,0)$ and is mapped to the imaginary axes. The period quotient map Γ^μ maps the upper half disk to a circular quadrilateral so that

$$0 \mapsto 0\ (\pi), \quad 1 \to 1\ (0), \quad -1 \mapsto -i\ (\pi/2).$$

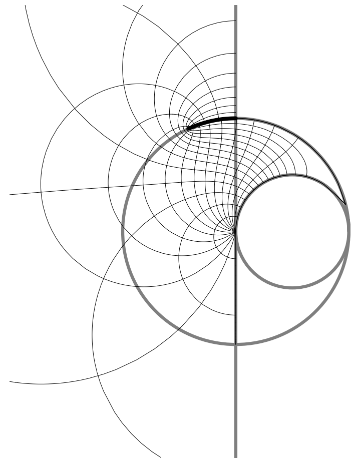
For $\mu < -4$ or $\mu > 0$, the branch points z_i are real, and one of them lies in the interval $[-1,1]$. The image of this branch point becomes a 2π vertex. For $\mu < -4$, this branch point is mapped to the lower imaginary axes and for $\mu > 0$ to the upper imaginary axes. For both B^μ and Γ^μ, the interval $[0,1]$ is mapped to the circle around $\frac{1}{2}$ with radius $\frac{1}{2}$, the half circle from 1 to -1 is mapped to the unit circle around 0, and the segment $[-1,0]$ is mapped to the imaginary axes.

Proof. The location of the additional branch points z_i follows from the formulas for the Schwarzian derivative in Corollary 4.4. If these branch points are real, the Schwarz-Christoffel formula implies that the upper half plane is mapped to a circular pentagon. By the symmetries of the period maps (Lemma 4.2), the upper half circle is mapped to the unit circle, and hence the pentagon is symmetric and can be obtained by reflecting a cirular quadrilateral at the unit circle. The angles at the vertices follow from corollaries 4.6, 4.7 and lemma 4.8, and the further symmetries imply the remaining statements. □

Remark 4.10. For other values of μ, the branch points z_i are located on the unit circle. By taking a standard map ϕ from the upper half plane to the upper half disk, the mapping behaviour of the composite maps $B^\mu \circ \phi$ and $\Gamma^\mu \circ \phi$ can then be completely analyzed for all values of μ. Below to the right is a figure showing the 2π vertex on moving on the unit circle in a B^μ image domain for $\mu > \frac{1}{4}$.



Γ^μ for $\mu = 0.43$



B^μ for $\mu = 1.0$

Remark 4.11. The same method allows to compute the angles of more general period quotients built from the $\alpha(a, b; \lambda)$. However, while the angles remain simple, the Schwarzian derivative as an intermediate result becomes very complicated, so we haven't included the results here.

There is a way to circumvent parts of this computation. Following [Th], one can view the period quotient map as the developing map of a projective structure on the moduli space of tori with marked 2-division points, or equivalently, on the moduli space of the 4-punctured sphere. The points in this moduli space can be interpreted as flat cone metric on the punctured sphere, where the cone angle is allowed to become negative to allow for funnel-like ends whose closed end has the corresponding *positive* cone angle. The behavior at the singularities can then be computed as in [Th] in terms of the cone angles. Note however that we need here more information than obtained from [Th]: We also need to *compare* different developing maps and therefore have to evaluate the developing map explicitly at special points, as done in Lemma 2.17, 2.18.

There is a process of passing from one projective structure to another by surgery called *grafting*, and our quadrilateral families B^μ, Γ^μ may be thought of as a continuous interpolation between the projective structures which arise when a quadrilateral degenerates to a triangle.

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