The Genus One Helicoid as a Limit of Screw-Motion Invariant Helicoids with Handles

(joint work with D. Hoffman and M. Wolf)

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Contents

1 Introduction

Besides the notorious plane, the only known complete embedded minimal surfaces of finite topology (that is: of finite genus and with finitely many ends) in euclidean space in the nineteenth century were the catenoid and the helicoid.

By a theorem of Collin ([Col97]), any such surface has either finite total curvature or just one end. The finite total curvature case is rather well developed: One has many examples (the Costa surface and its descendants) and many interesting conjectures (see besides others [Cos82, Cos84, Cos89, HMI85, HMI87, HMI89, HMI90, Kap97, Kar89, Tra98, Tra01, WW98, WW02, Woh91, Woh93]).

In the 1-ended case, there is only one new example so far, the Genus One Helicoid ([HKW93a, HKW93b]). Its existence proof (Hoffman, Karcher, Wei) was very complicated, and for the embeddedness one had to believe the computer graphics by Jim Hoffman.

However, Hoffman and Wei ([HW]) outlined another, more conceptual approach. They proposed the existence of a family of screw motion invariant minimal surfaces with a helicoidal end and periodic handles. This family should contain both the translation invariant helicoid with handles, a surface known to exist and to be embedded ([HKW99]), and, as a limit, the genus one helicoid. Using the maximum principle and known results about helicoidal ends ([HM99, HPR99]), the embeddedness of the genus one helicoid would follow.

They derived the Weierstrass data for this family and provided numerical and graphical evidence that this family indeed exists. However, they were not able to solve the period problem at this time.

We will here present joint work of Hoffman, Wolf and myself ([WHW01]) which is based on the original idea of Hoffman and Wei, the cone metric technique developed by Wolf and myself, and my Habilitation thesis ([Web00]) which proves both the existence of the proposed family and the embeddedness of the genus one helicoid.

In the first part, we will give a short complete proof of the existence of the translation invariant helicoid with handles, as this serves as the model case for the general case, which will be sketched in the second part.

Bloomington, October 2001 — Matthias Weber

2 The translation invariant helicoid with handles

Theorem 2.1 *There exists a complete embedded minimal surface $\mathcal{H}_1$ in $\mathbb{R}^3$ with the following properties:*

1. *The surface is invariant under a vertical translation.*
2. *The quotient of the surface by the translation has genus 1.*
3. *The quotient surface has two helicoidal ends.*
4. *The vertical coordinate axis lies on the surface and is a symmetry line.*
5. *The quotient surface has two parallel horizontal lines crossing the vertical axis which are lines of (the same) symmetry.*

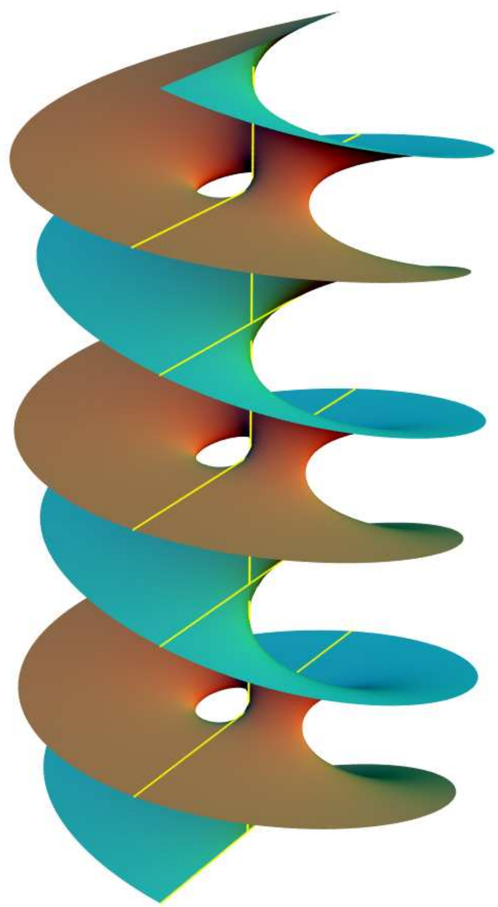

Figure 1: The Translation Invariant Helicoid With Handles

2.1 The Weierstrass data

The quotient surface can be represented by a torus $\mathcal{T}_1$ punctured at two points corresponding to the two helicoidal ends. This torus is necessarily rhombic, as the required symmetries have connected fixed point sets.

At the ends, the height differential must have simple poles with purely imaginary residues at the ends. To locate the zeroes, note that these will be permuted by rotations around the vertical or a horizontal axis. As there are only two and they can't lie on the vertical axis, they must both lie on one of the horizontal axes. At these points, the Gauss map is vertical.

Normalize a fundamental domain for $\mathcal{T}_1$ to have vertices at $\pm 1, \pm\tau i$ so that the vertical diagonal represents the vertical straight line and the horizontal diagonal represents the horizontal straight lines. Using symmetry, we can then write

$$\begin{aligned} V_1 &= -a & \qquad V_2 &= a \\ E_1 &= -b & \qquad E_2 &= b \end{aligned}$$

with $a < b < 1$. Abel's theorem applied to G forces

$$a + b = 1$$

Divisor table for $\mathcal{H}_1$:

	E_1	V_1	V_2	E_2
dh	∞	0	0	∞
G	∞	∞	0	0
Gdh	∞^2	$*$	0^2	$*$
$\frac{1}{G}dh$	$*$	0^2	$*$	∞^2

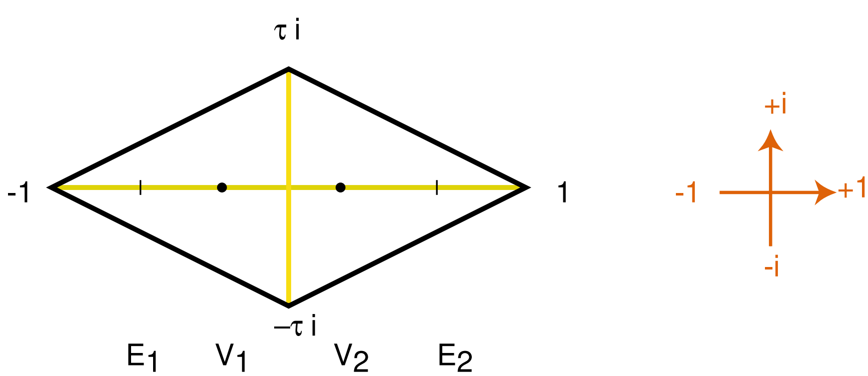

Figure 2: Divisor of the Weierstrass data

The horizontal diagonal of $\mathcal{T}_1$ corresponds to the two horizontal straight lines and the vertical diagonal to the vertical line of $\mathcal{H}_1$. We will encounter a slightly confusing technical problem in these figures: The imaginary direction in the complex plane does not always correspond to the vertical direction in the 3-dimensional euclidean space where the surface is sitting. To avoid any misunderstandings, we have included compasses in each planar figure in this section.

2.2 The horizontal period condition

The horizontal period condition requires that

$$\int_\alpha G dh = \overline{\int_\alpha \frac{1}{G} dh}$$

for all closed cycles α on the torus $\mathcal{T}_1$.

This is equivalent to the condition that the periods of the first two complex coordinate differentials are purely imaginary, see the authors general notes in the same volume.

Consider the involution $\iota : z \mapsto -z$. It carries the divisor of Gdh to the divisor of $\frac{1}{G}dh$, and as the differential at a fixed point is -1 we have

$$\iota^* \frac{1}{G} dh = -G dh$$

Using the period condiion it follows that

$$\begin{aligned} \int_\alpha G dh &= -\int_\alpha \iota^* \frac{1}{G} dh = -\int_{\iota(\alpha)} \frac{1}{G} dh \\ &= \int_\alpha \frac{1}{G} dh = \overline{\int_\alpha G dh} \end{aligned}$$

so that all the periods of Gdh need to be purely real.

Slit the plane from -1 to 1, and glue the $[-1, 0]$ portion of the upper (resp. lower) part of the slit to the $[0, 1]$ portion of the lower (resp. upper) part of the slit.

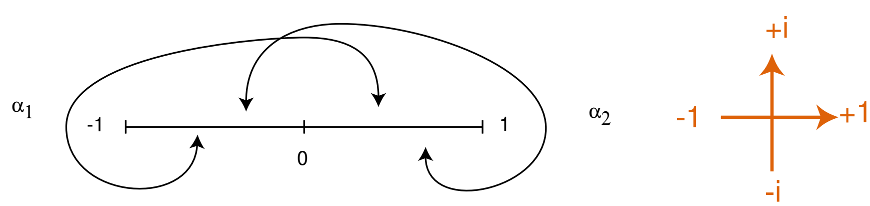

Figure 3: Cone metric construction of the rhombic torus and the meromorphic 1-form

This defines a cone metric on a rhombic torus with trivial holonomy and two cone points. At the point $\{-1, 0, 1\}$, the cone angle is 6π, and at ∞, the cone angle is -2π. Hence the cone metric comes from a 1-form with double order pole and zero and only real periods.

We identify this cone metric torus with a rhombic lattice torus $\mathcal{T}_1$ so that the imaginary axis of the slit plane becomes the horizontal diagonal of the rhombus. We translate the cone metric along the horizontal diagonal so that the 6π cone point coincides with $V_2 = a$. This defines Gdh. Finally, define $\frac{1}{G}dh = -\iota^* Gdh$.

It is conceivable that there are other tori and forms Gdh with the desired conditions. On a rhombic toris one can show that this is in fact the only choice, see [Web02].

Using the jigsaw puzzle in figure 4,

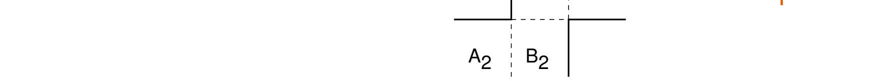

Figure 4: A jigsaw puzzle

we can decompose the cone metric into two planar domains. In figure 5 the cycles α_1 and α_2 from figure 3 are included.

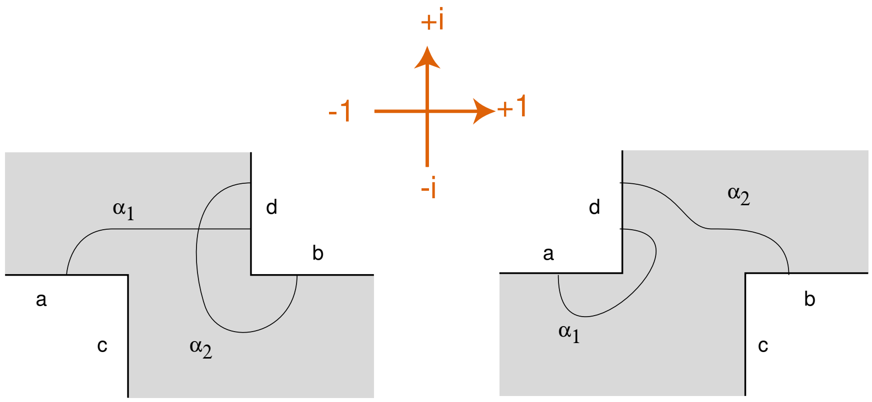

Figure 5: Alternative description of the cone metric

2.3 The vertical period condition

As we have defined Gdh and $\frac{1}{G}dh$ on a fixed torus $\mathcal{T}_1$, dh is also defined up to sign. The only freedom we are left with is to vary the parameter a.

We first check that the residues of dh at E_1 and E_2 are purely imaginary. This is necessary so that the ends at E_1 and E_2 are helicoidal. Consider a positive real tangent vector V at the center 0 of the rhombic fundamental domain. By definition, $Gdh(V)$ is positive imaginary, and $1/Gdh(V) = -\iota^* Gdh(V) = Gdh(V)$ is as well. Hence

$$dh(V) = \sqrt{Gdh(V) \cdot 1/Gdh(V)} \in i\mathbb{R}$$

at 0. Now note that dh is odd, as the divisor is invariant under ι and it can't be even, as it then would descend to the quotient sphere. Therefore this condition holds on the whole horizontal diagonal. Hence the residue of dh at the ends E_1 and E_2 is purely imaginary, as desired.

We now choose the sign of dh so that at 0,

$$dh(V) \in i\mathbb{R}^-$$

The vertical period condition requires for the cycle γ in figure 6

$$\operatorname{Re} \int_\gamma dh = 0$$

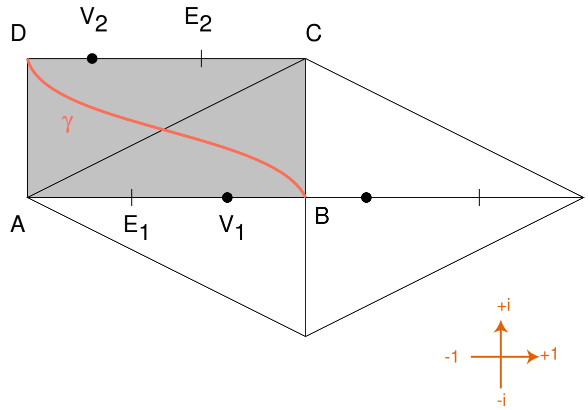

Figure 6: Vertical period condition

We call the shaded rectangle the fundamental rectangle. Its boundary edges are the straight lines on the surface. The fundamental rectangle is mapped by the surface parameterization to a surface patch which looks as in figure 7. The path γ is mapped to a closed curve connecting B with D around the handle.

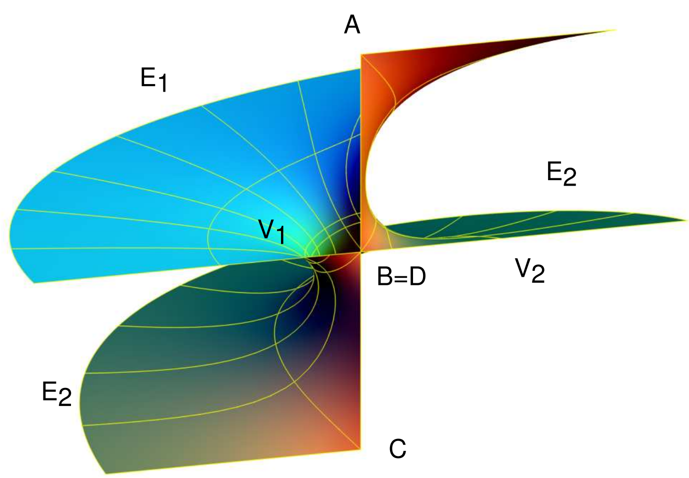

Figure 7: Image of the fundamental rectangle

Proposition 2.1 *The periods of dh are continuous in a for $a \in [0, \frac{1}{2}]$.*

Proof: The argument requires that we spend a few lines on the meaning of the word continuous. For the spaces of meromorphic 1-forms and functions on a fixed torus, we will use the topology of locally uniform convergence. This topology has the property that translations of forms, products, other standard arithmetic operations and integrals over cycles avoiding singularities are continuous. It is by no means the only toplogy with this property, but it is quite convenient.

Now, the forms Gdh and $\frac{1}{G}dh$ depend on a only by a translation and hence make sense for all a. By the definition of dh as

$$dh = \sqrt{Gdh \cdot \frac{1}{G}dh}$$

it follows that also dh depends continuously on a for all a, and therefore also the periods of dh depend continuously on a. □

Proposition 2.2 *For $0 < a < \frac{1}{2}$, the map $\int^z dh$ maps the fundamental rectangle to a polygonal domain. The positive real axis points up in the dh figures so that up in the figure means up in space. The edges emanating from V_1 and V_2 are slits, and the other horizontal edges extend to infinity to the left and right, so that the region has two infinite half-strips as ends.*

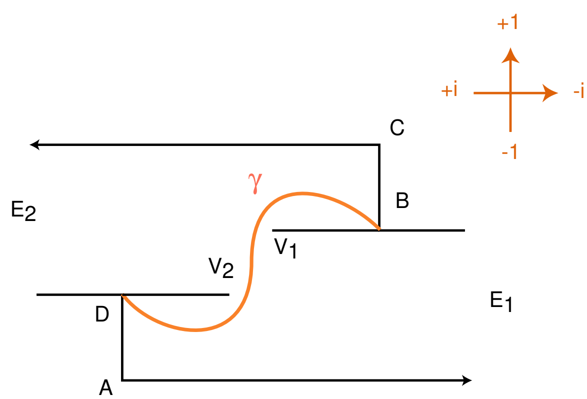

Figure 8: Image of the fundamental rectangle under $\int dh$

Proposition 2.3 *For $a = \frac{1}{2}$, $\int^z dh = \lambda iz + c$ with $\lambda \in \mathbb{R}^-$*

Proof: In this case, $a = b = \frac{1}{2}$ so that Gdh and $\frac{1}{G}dh$ cancel to $dh = \lambda idz$ (recall that we defined $\frac{1}{G}dh = -\iota^* Gdh$). □

The figure 9 shows the image of $\int^z dh$ for $a = 0.49$.

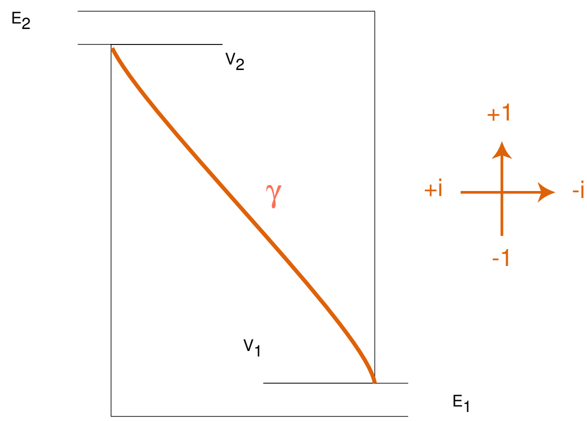

Figure 9: Vertical period condition

Proposition 2.4 *For $a = 0$, $\int^z dh$ maps the fundamental rectangle to the domain in figure 10.*

Proof: In this case, $b = 1$ so that $Gdh = -\frac{1}{G}dh$ and hence $dh = iGdh$. At this point the referee complained that one can deduce $G \equiv -i$ i the limit case. This is indeed the case, but doesn't help at all — we need to determine boundary values for period integrals to be able to apply the implicit function theorem.

In this case, the form $iGdh$ is symmetric with respect to the rhombus diagonals, and hence the fundamental rectangle is mapped to one of the (rotated) pieces of the jigsaw puzzle (figure 4). because it decomposes into these two pieces under the same symmetries. It must be the piece in figure 10, because, by construction, the edge E_1V_1 must point to the left, and this rules out the other jigsaw piece. □

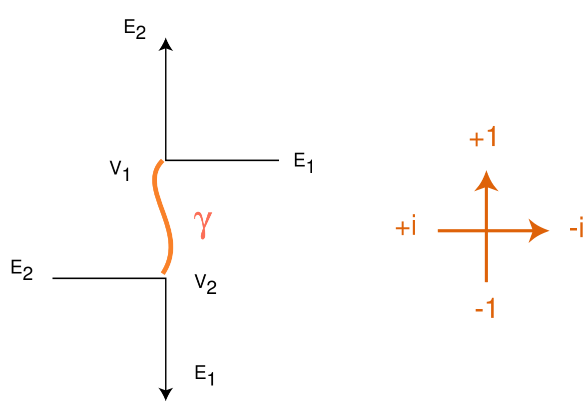

Figure 10: Limit domain for dh at $a = 0$

From the intermediate value theorem we get

Theorem 2.2 *There is a value $0 < a < \frac{1}{2}$ which solves the vertical period problem.*

This concludes the existence proof of the translation invariant helicoid with handles.

Remark 2.1 The embeddedness of this surface was proven first in [HKW99]. There is now an alternative proof by F. Martín which uses the above defomation of the fundamental piece in the parameter a.

3 The screw-motion invariant helicoids with handles

In this section, we modify the proof for the existence of the translation invariant helicoid with handles to screw-motion invariant helicoids with handles. One of the key advantages of the cone metric approach is that multivalued forms can be treated easily.

The main additional difficulty in this section comes from the fact that we have to allow the underlying torus to vary as well as the 1-forms. This makes it necessary to be very careful with the topology on the parameter spaces. We will be rather sketchy in the argumets, however, and refer the reader to the forthcoming paper [WHW01] for the full details.

We begin by reviewing conditions on the Weierstrass data for screw-motion invariant minimal surfaces in general.

3.1 Singly periodic minimal surfaces invariant under a screw-motion

Suppose Σ is a complete minimal surface in $\mathbb{R}^3$ invariant under a vertical screw motion with rotation angle ϕ and translation t. Let X be the quotient surface. Cutting Σ with a horizontal plane defines a homology class $\eta \in H_1(X, \mathbb{Z})$.

Proposition 3.1 *The Weierstrass data $(dG/G, dh)$ are well-defined on X and satisfy*

1. $\int_\gamma \frac{dG}{G} = \#(\gamma, \eta) \cdot \phi$
2. $\int_\gamma dh = \#(\gamma, \eta) \cdot t$
3. $\int_\gamma G dh = \overline{\int_\gamma 1/G dh} \qquad \forall \#(\gamma, \eta) = 0$

If, on the other hand, Weierstrass data $(dG/G, dh)$ satisfy these conditions, they define a screw motion invariant surface as stated.

Here is a list of the known examples of screw-motion invariant minimal surfaces:

1. Karcher ([Kar88, Web00]): Twisted Scherk Surfaces
2. Hoffman-Karcher-Wohlrab ([Lyn93]): Twisted Fischer-Koch surfaces
3. Callahan-Hoffman-Karcher ([CHK93]): Twisted Callahan-Hoffman-Meeks surfaces
4. Hoffman-Karcher-Wei ([HKW93a, HKW93b, HKW99, HW]) and Weber-Hoffman-Wolf ([Web00, WHW01]: Twisted Helicoids with Handles
5. Traizet-Weber ([TW01]): Many new examples, among them helicoids with arbitrarily many handles in the quotient

3.2 The screw-motion invariant helicoids with handles

Theorem 3.1 *For any angle $\pi < \phi < \infty$, there exists a complete embedded minimal surface $\mathcal{H}_k$ in $\mathbb{R}^3$ with the following properties:*

1. *The surface is invariant under a vertical screw motion with angle $\phi = 2\pi k$ and translation t.*
2. *The quotient of the surface by the screw motion has genus 1 and two helicoidal ends.*
3. *The vertical coordinate axis lies on the surface.*
4. *The quotient surface has two horizontal lines crossing the vertical axis.*
5. *For the twist angle we have $\phi = \int_\nu \frac{dG}{G}$ where ν is a segment of the vertical coordinate axis of length t.*

To prove this theorem, we begin by analyzing the divisors of the Weierstraß data. The multivaluedness of the Gauss map is visible at the real exponent k at the ends.

The divisors of G and dh are given by

	E_1	V_1	V_2	E_2
dh	∞	0	0	∞
G	∞^k	∞	0	0^k
Gdh	∞^{k+1}	$*$	0^2	0^{k-1}
$\frac{1}{G}dh$	0^{k-1}	0^2	$*$	∞^{k+1}

Instead of using dG/G to define G, we construct a cone metric for $|Gdh|$ as follows: Take the slit torus domain as for the translation invariant helicoid with handles, choose a point hi on the positive imaginary axis, slit the positive imaginary axis above this point to $i\infty$ and sew in a cone with cone angle $2\pi(k-1)$ so that the total cone angle at hi becomes $2\pi k$. This cone metric defines a multivalued 1-form Gdh.

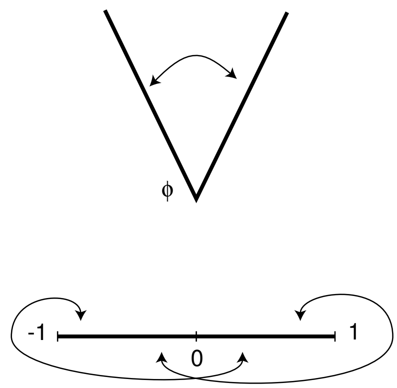

Figure 11: Cone metric for Gdh

For each $(k,h) \in K = (\frac{1}{2}, \infty] \times [0,\infty]$, we have defined a cone metric $|Gdh|$. Note that for $h=0$ the construction makes still sense even though cone points have coalesced. For $h=\infty$, the whole cone has disappeared. It will be of course of great importance to ensure that further constructions are continuous in (k,h) on the full set K.

We identify the cone metric torus with a rhombic torus $\mathcal{T}_{k,h}$ given by a rhombus with corners at $\pm 1, \pm i\tau$ so that the vertical axis becomes the horizontal rhombus diagonal and so that the divisor becomes symmetric at the vertical rhombic diagonal. We can arrange so that

$$\begin{aligned} V_1 &= -a & \qquad V_2 &= a \\ E_1 &= -b & \qquad E_2 &= b \end{aligned}$$

Then $\frac{1}{G}dh$ is defines as $-\iota^* Gdh$ with $\iota(z) = -z$.

For $k=\infty$, the same construction works as long as we realize that a cone with infinite cone angle represents only a single cone point. This means $E_1 = E_2$ and hence $b = 1$ in the construction. We will come back to this later.

In any case, this pair of cone metrics defines a pair of of multivalued 1-forms with the right monodromy so that the horizontal period condition is satisfied.

In this screw-motion invariant situation, the solution to the horizontal period problem does not define the underlying torus uniquely. We find a 1-parameter family of candidates $\mathcal{T}_{k,h}$ where $h \geq 0$. However, the parameter k is the only free parameter left, because it specifies the position of the cone point E_2 in addition to V_2 and E_1. Hence the divisor of dh is completely determined (by symmetry).

3.3 An Abel theorem

Proposition 3.2 *Suppose there is a family of screw-motion invariant surfaces $\mathcal{H}_k$ as in theorem 3.1 so that $\mathcal{H}_1$ is the translation invariant helicoid with handles. Then necessarily*

$$a + kb = k$$

Proof: Apply the reciprocity formula to the forms dz and $\frac{dG}{G}$: Let R be a rhombic fundamental domain of $\mathcal{T}_{k,h}$ with corners at ± 1 and $\pm i\tau$. Then, by the residue theorem,

$$\int_{\partial R} z \cdot \frac{dG}{G} = 4\pi i(ka + b)$$

On the other hand, by direct computation

$$\int_{\partial R} z \cdot \frac{dG}{G} = (1 + i\tau)2\pi k - (1 - i\tau)2\pi k = 4\pi ik$$

Here we use that, along the boundary edges, G changes its value by $e^{\pm 2\pi ik}$. This is deliberately, but this is the only choice which is continous in k and holds with the translation invariant case ($k = 1$). □

3.4 Continuity of the vertical period

Let γ be the cycle connecting 0 with $-1 + \tau i$ diagonally in the fundamental rectangle. Let $K = (\frac{1}{2}, \infty] \times [0, \infty]$ be the parameter rectangle. Then define

$$\begin{aligned} H : K &\rightarrow \mathbb{R} \\ H(k, h) &= \operatorname{Re} \int_\gamma dh \end{aligned}$$

Proposition 3.3 *H is continuous in K and real analytic in $(\frac{1}{2}, \infty) \times (0, \infty)$.*

Proof: The real analyticity in the interior of K is clear as H is given by the real part of a composition of holomorphic period maps and inverse period maps.

Hence we need to understand continuity only at the boundary. We distinguish two cases:

Suppose first that h is large. Decompose the $|Gdh|$ cone metric into two domains:

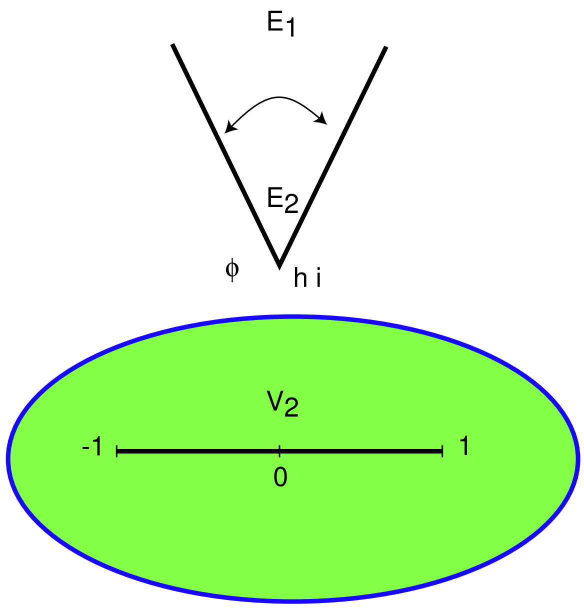

Figure 12: Domain decomposition

These domains correspond to two domains in $\mathcal{T}_{k,h}$:

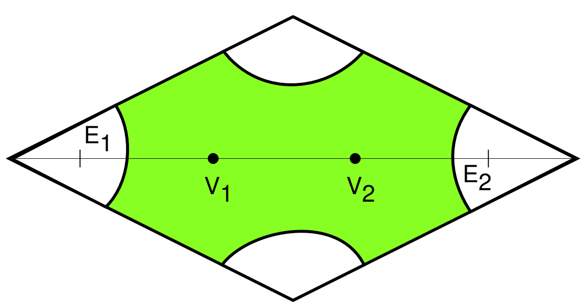

Figure 13: Domain decomposition

The white region with the $|Gdh|$ cone metric is isometric to the complement $\Omega = \Omega(k, h)$ of a neighborhood of 0 in the complex plane with a cone metric given explicitly by

$$
\begin{aligned}
ds &= \left| d\left(1+\frac{z}{kh}\right)^k \cdot h \right| \\
&= \left| \left(1+\frac{z}{kh}\right)^{k-1} dz \right|
\end{aligned}
$$

This metric has cone points at $-kh$ and ∞, and the point $-kh$ is at distance h from 0, as

$$
\int_{-kh}^{0} d\left(1+\frac{z}{kh}\right)^k \cdot h = \left(1+\frac{z}{kh}\right)^k \cdot h \Big|_{-kh}^{0} = h
$$

To understand the convergence of the cone metric $|Gdh|$, it suffices to understand the convergence of the metric ds and of the gluing maps between the cone metric in Ω and the green region in $\mathcal{T}_{k,h}$. The gluing maps are defined in an overlap region of the green and white domains.

Now for $h \to \infty$, $ds \to |dz|$ uniformly in k and locally uniformly in z. Similarly, for $k \to \infty$, $ds \to |e^{z/h} dz|$.

In particular, the convergence is uniform in the compact gluing region. These statements suffice to conclude that the 1-forms Gdh (and hence $\frac{1}{G}dh$, dh and $H(k,h)$) converge locally uniformly for (k,h) converging to the upper and right boundary of K.

For $h \to 0$, one uses another decomposition of the cone metric. This is topologically more complicated, but the convergence is simpler as only finite cone points coalesce. Details can be found in [WHW01].

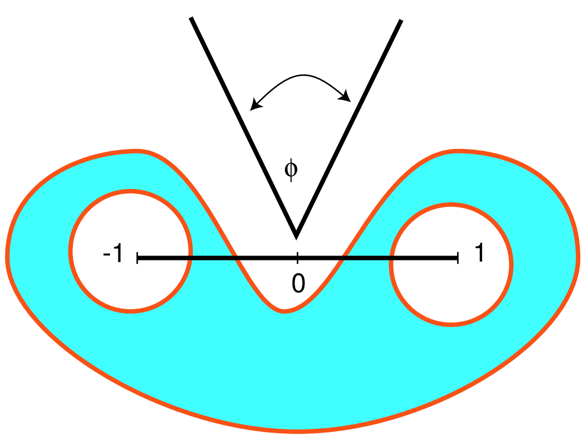

Figure 14: Domain decomposition

□

Using the continuity of H, one proves the existence of an $\mathcal{H}_k$ exactly in the same way as for the translation invariant $\mathcal{H}_1$:

The following two propositions are proved in exactly the same way as propositions 2.3 and 2.4 in the translation invariant case.

Proposition 3.4 *For $h = 0$, $\int^z dh = \lambda i z + c$.*

This gives by the intermediate value theorem:

Proposition 3.5 *For $h = \infty$, the cone metric $|Gdh|_\infty$ is the same as for $\mathcal{H}_1$ and hence $\int^z dh = \int Gdh_\infty$.*

Theorem 3.2 *For any $k > \frac{1}{2}$, there is a value $0 < h < \infty$ (or $\frac{k}{k+1} < b < 1$) such that the vertical period problem for the twisted genus 1 helicoid (resp. the genus one helicoid, if $k = \infty$) is solved.*

Remark 3.1 The method breaks down for $k \leq \frac{1}{2}$, as then the limit case $h = 0$ cannot be realized anymore. There is strong theoretical ([TW01]) and numerical evidence that the $k = \frac{1}{2}$ surface does not exist.

4 The Genus One Helicoid

4.1 Existence

Theorem 4.1 *There exists a complete embedded minimal surface in $\mathbb{R}^3$ with the following properties:*

1. *The surface has genus 1 and infinite total curvature.*
2. *The surface has one helicoidal end.*
3. *The vertical coordinate axis lies on the surface and is a symmetry line.*
4. *The Gauss map has an essential singularity at the end.*

5. *The surface has one parallel horizontal line crossing the vertical axis.*

The existence part of the proof is very similar to the one for the twisted helicoids: For the $|Gdh|$ cone metric, one uses a cone with an infinite cone angle. The height differential has a double order pole at the end and two simple zeroes at the finite points where the Gauss map is vertical. These zeroes can coalesce either at the end or at the center of the torus. These limit cases correspond again to explicitly known non-solutions of the vertical period problem with opposite signs. Details can again be found in [WHW01].

4.2 Continuity of the family

Theorem 4.2 *There is a continuous family of twisted helicoids* $\mathcal{H}_k$ *for* $\frac{1}{2} < k_0 \le k \le \infty$.

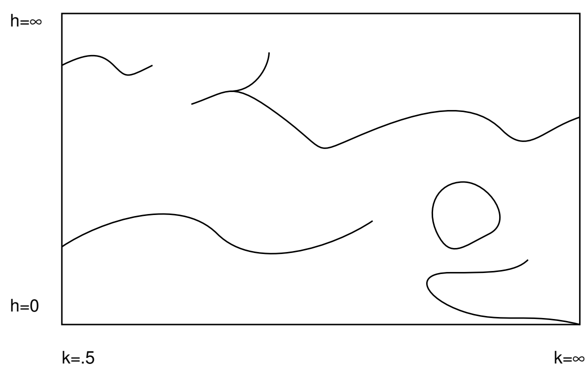

Figure 15: The solution locus?

Proof: Fix $k_0 > \frac{1}{2}$, and denote the solution locus by

$$Z = \{(k,h) \in [k_0,\infty] \times [0,\infty] : H(k,h) = 0\}$$

For every k, there is a $h(k)$ such that $(k,h(k)) \in Z$. Suppose there is no connected component of Z containing points with $k = k_0$ and $k = \infty$. Then Z could be separated by an arc connecting the horizontal edges $h = 0$ and $h = \infty$ which are disjoint from Z. However, H is continuous along that arc and takes positive and negative values at the endpoints by propositions 3.4 and 3.5. Hence there is a connected component Z_0 of Z connecting a solution at k_0 with a solution at $k = \infty$. As H is real analytic, Z is locally path connected and hence Z_0 is path connected. □

Remark 4.1 We conjecture that this family is smooth and unique. Note however that numerical computations of Bobenko ([Bob]) indicate that there is a non-embedded genus one helicoid.

4.3 Embeddedness of the Genus One Helicoid

Theorem 4.3 *The twisted helicoids $\mathcal{H}_k$ of the continuous family are embedded.*

Proof: We know that any $\mathcal{H}_1$ is embedded. Scale and arrange the $\mathcal{H}_k$ in $\mathbb{R}^3$ so that

1. All $\mathcal{H}_k$ are asymptotic to the same helicoid as $\mathcal{H}_\infty$.
2. The straight line containing V_1 and V_2 is the x-axis.
3. The centers of the rhombi are mapped to the origin.

As the helicoidal end is embedded ([HM99, HPR99]), the maximum principle implies that all $\mathcal{H}_k$ are embedded. □

Remark 4.2 One can easily arrange the $\mathcal{H}_k$ so that they converge to the ordinary helicoid. We avoid this by choosing a common (geometric) reference point of all surfaces. As we have continuity of the family including infinity, this gives convergence of the geometric surfaces in $\mathbb{R}^3$.

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