Construction of Harmonic Surfaces with Prescribed Geometry

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Abstract. In this note we explain how a well-understood construction method for minimal surfaces can be used as flexible tool to explicitly parametrize harmonic surfaces with prescribed geometry of arbitrary finite topological type.

1 Introduction

A fundamental problem in surface geometry has been the construction of surfaces with given local properties like constant curvature and prescribed global geometric features. In the case of minimal surfaces, this has reached a state where amazingly complex surfaces can be constructed using relatively elementary means. Furthermore, the availability of rather explicit parametrizations has allowed to numerically and visually explore phenomena that have furthered the theory in a way unconceivable 10 years ago.

This success in the realm of minimal surfaces is mostly due to the availability of the Weierstrass representation, which is limited to minimal surfaces (and a few other related situations).

Thus, if one wants to use these techniques as a general modeling tool, the limitation to minimal surfaces limits also the topological possibilities. For instance, by a theorem of Schoen [3], there is no catenoid with a handle, i.e. no embedded, complete minimal surface with two ends and genus one. To overcome this limitation and still be able to make use of the machinery developed for minimal surfaces, we suggest to consider the class of harmonic surfaces where the parametrization is given by a harmonic map into $\mathbb{R}^3$.

In this note, we will illustrate how to implement this idea using a concrete example.

2 The Minimal Catenoid with a Handle

Definition 1. *Let Ω be a Riemann surface, $G(z)$ a meromorphic function and $dh(z)$ a holomorphic 1-form on Ω. Then*

$$f(z) = \operatorname{Re} \frac{1}{2} \int^{z} (1/G - G, i/G + iG, 2)\; dh \tag{1}$$

parametrizes a minimal surface away from the singularities of G and dh. This parametrization is conformal.

The most simple instance of applying the Weierstrass representation to construct concrete minimal surfaces is the

Lemma 1. *Let Σ be a minimal surface, conformally equivalent to the disk, so that the boundary of $\bar{\Sigma}$ consists of finitely many planar symmetry curves that lie in vertical planes. Assume that all points with vertical normal occur only at intersection points of the boundary and possibly at the ends of the surface. Then Σ can be parametrized using a Gauss map $G(z)$ and a height differential dh that are given as*

$$G(z) = \rho \prod (x - a_j)^{\alpha_j} \tag{2}$$

$$dh = \mu \prod (x - a_j)^{\beta_j}\, dz \tag{3}$$

where the a_j are points on the real axis in one-to-one correspondence to the vertices of the surface boundary, the α_j, and β_j are real numbers uniquely determined by the angles between the symmetry lines at the vertices, ρ is the López-Ros parameter, and μ a scale factor.

For more sophisticated applications and a proof, see [1].

To construct an (impossible) catenoid with a handle, let's assume symmetries at all three coordinate planes (see [2]). The two vertical coordinate planes then cut the surface into four congruent simply connected pieces. The points with vertical normal occur at the catenoidal ends C_1 and C_2, and at two points H_1 and H_2 in the handle. Let's focus on the piece in which the boundary curves meet counter clockwise in the points C_1, C_2, V_2, V_1 in this order. Using the horizontal symmetry and a Mobius transformation of the upper half plane, we can assume that these points correspond to ∞, 0, $1/a$, a for some real number $a > 1$. Observe that the assumed symmetry about the horizontal coordinate plane is realized by a reflection at the unit circle in the upper half plane. Then it is easy to see that by

$$\begin{aligned} G(z) &= \frac{1}{\sqrt{a}} z^{-1/2} (z - 1/a)^{-1/2} (z - a)^{1/2} \\ dh &= dz/z \end{aligned}$$

any catenoid with a handle and the assumed symmetry properties would be given by this formula. The problem is that no matter how one tries to adjust the parameter a, it is impossible to close all periods: The two boundary curves between C_1 and C_2 and between V_1 and V_2 are supposed to lie in plane $x_1 = 0$, but the formula only guarantees that they lie in parallel planes, see the left image in Figure 1

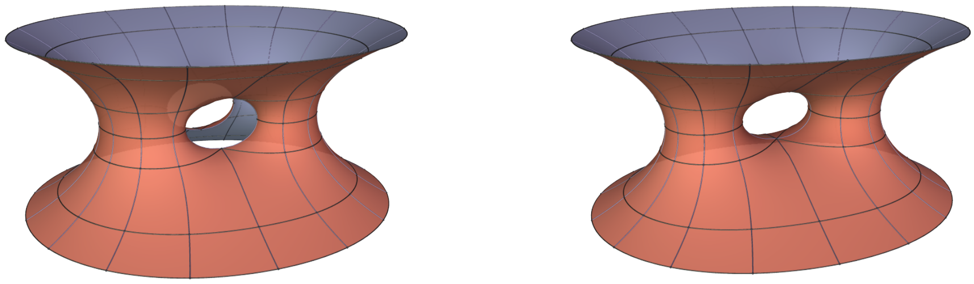

Fig. 1. Minimal Catenoid with Handle and Unclosed Period and Harmonic Catenoid with a Handle

3 The Harmonic Catenoid with a Handle

To understand how it is possible to *canonically* overcome this problem, it helps to switch to the global picture for a moment: A catenoid with a handle would be defined on a rectangular torus, where the Weierstrass data $G(z)$ and dh have become a meromorphic function and 1-form. The representation formula 1 parametrizes a well-defined surface if and only if *all* the periods of the forms ω_j are purely imaginary. This is not the case, as ω_1 has still a real period (while both other forms have purely imaginary periods). This can be remedied by adding a suitable multiple of the holomorphic 1-form dz of the torus to ω_1. In fact, this problem can be corrected uniquely on any compact Riemann surface in a unique way, as the real parts of its periods determine a holomorphic 1-form uniquely. This uniqueness guarantees that all assumed symmetries of the putative minimal surface will persist for the harmonic surface. Moreover, as we only modify the surface by adding a holomorphic form, the asymptotic behavior of the surface won't change. In our case, this means that the surface will still have ends asymptotic to a catenoid. This shows that for any $a > 1$, there is a unique harmonic catenoid with a handle and with all symmetry properties as claimed.

More generally, using the same approach, we can prove:

Theorem 1. *Let Σ be an immersed surface, diffeomorphic to a disk, so that the boundary of $\bar{\Sigma}$ consists of finitely many symmetry curves that lie in vertical planes. Assume that all points with vertical normal occur only at intersection points of the boundary and possibly at the ends of the surface. Also assume that reflection at the vertical symmetry planes extends the surface to a surface $\hat{\Sigma}$ of finite topological type whose ends are asymptotic to catenoidal or planar ends. Then there is a harmonic surface $\tilde{\Sigma}$ whose boundary is isotopic to that of Σ in each symmetry plane. It is given by a Gauss map $G(z)$ and a height differential dh as in equation (2) using a modified Weierstrass representation $\tilde{\omega}_j = \omega_j + \phi_j$, where the ϕ_j extend to holomorphic 1-forms $\tilde{\omega}_j$. Any choice of the parameters a_j determines the ϕ_j uniquely.*

References

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