# A TEICHMÜLLER THEORETICAL CONSTRUCTION OF HIGH GENUS SINGLY PERIODIC MINIMAL SURFACES INVARIANT UNDER A TRANSLATION

Matthias Weber

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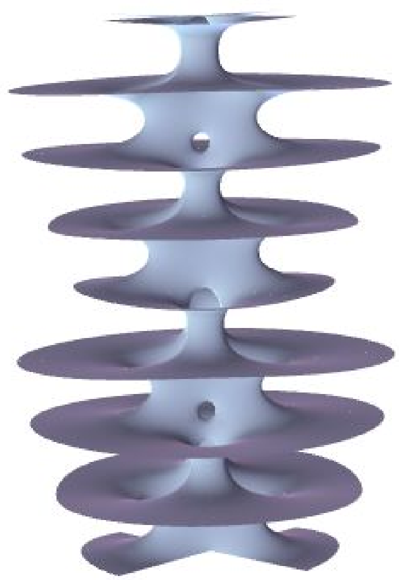
Abstract. We prove the existence of singly periodic minimal surfaces invariant under a translation such that a fundamental piece has arbitrarily many parallel planar ends and arbitrarily high genus. These surfaces generalize the Callahan-Hoffman Meeks surface. We also discuss briefly the effective computation of the periods and techniques to parameterize these surfaces.



Two fundamental pieces of $CHM_{2,3}$

## 1. Introduction

The known singly periodic embedded minimal surfaces which are invariant under a parallel translation are classified by their ends ([MR]), and the basic examples

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are:

(1) Scherk's saddle tower ([Sch])
(2) The Helicoid ([Meu])
(3) The Riemann minimal surface ([R])
(4) The Callahan-Hoffman-Meeks surface ([CHM])

All these surfaces allow for variations, like adding handles, increasing the dihedral symmetry, or twisting the surface so that the surface becomes invariant under a screw motion. See [CHK] for a detailed discussion of these variations, and also [Ka1, HKW1, HKW2].

However, when dividing one of these surfaces (or their known modifications) by its maximal translational symmetry group, the resulting surface has always small topology. In this paper we prove the existence of complete simply periodic minimal surfaces in $\mathbb{R}^3$ such that a fundamental piece of the surface by its symmetries has arbitrarily many ends and arbitrarily high genus. These surfaces are derived from the Callahan-Hoffman-Meeks surface by periodically adding handles at appropriate places, as partially suggested by numerical experiments carried out by Hoffman and Wohlgemuth (see [HK]). There is also some numerical evidence that our new surfaces are embedded.

This is a third paper in a series ([WW1,WW2]) using Teichmüller theory to deal with high-dimensional period problems. The surfaces here correspond to the surfaces $DH_{m,n}$ of finite topology in [WW2] in a very natural way, and the existence proof here is also a direct adaptation of the existence proof there. We take this opportunity to illustrate the flexibility of the method by giving a detailed overview.

The $DH_{m,n}$ surface has finite topology, $2m+1$ parallel planar ends and two catenoid ends. The genus can be any number $m+n+1$ if only $n \geq m$. Suitably arranged in $\mathbb{R}^3$ the surface has the two vertical coordinate planes as reflectional symmetry planes and the $y = x$ diagonal as a rotational symmetry line lying on the surface.

A fundamental quotient piece of the new surface $CHM_{m,n}$ is roughly obtained from $DH_{m,n}$ by removing the top and bottom catenoidal ends and identifying the top and bottom planar ends. The symmetry group will remain the same, it is generated by a twofold dihedral symmetry at the vertical coordinate planes and a rotational symmetry around the $y = x$ diagonal.

Thus our goal will be the

**Main Theorem.** *For each pair of integers $1 \leq m \leq n$, there is a singly periodic complete minimal surface $CHM_{m,n}$ in $\mathbb{R}^3$ invariant under a translation with the symmetries described above, whose quotient by the translational symmetry group has $2m$ planar ends and genus $m+n+1$.*

Besides the existence of such high genus singly periodic minimal surfaces, this discussion of the surfaces is motivated by the question how sequences of $DH_{m,n}$ surfaces might evolve for high values of $m$ and $n$. In fact we expect the sequence $DH_{km,kn}$ to converge in some sense to the surface $CHM_{m,n}$ for $k \to \infty$. Furthermore, to gain intuition about possible further modifications of these surfaces, it was desirable to have pictures available.

In section 2, we will give a description of the new surfaces in terms of their euclidean and orthodisk geometries. In sections 3 and 4, we will outline the general

Teichmüller theoretic existence strategy and give the necessary modifications in the proof. In section 5 we will explain how the period integrals can be transformed algebraically to allow the computation of the periods as real line integrals, facilitating the numerical computations. We will also show how to set up coordinate systems for the surfaces to obtain parameterizations which are well adapted to the ends and allow to create acceptable pictures.

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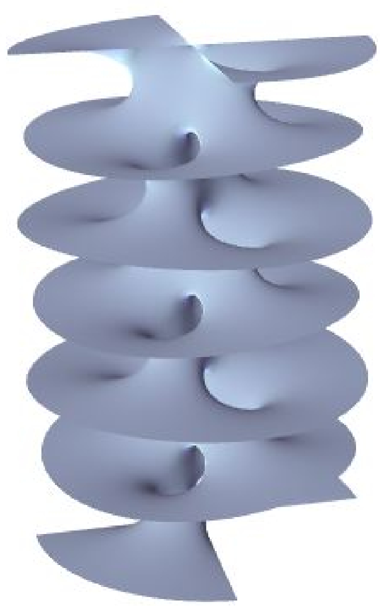
## 2. Description of the Surfaces.

**The Callahan-Hoffman-Meeks surface.**

We begin this section by a description of the well-known Callahan-Hoffman-Meeks surface (see also [CHM] or [MaRo]) to fix our notation.

This surface, denoted here by $CHM_{1,1}$, has infinitely many parallel planar ends. It can be arranged in 3-space so that these ends are parallel to the $x$-$y$-plane and that the other two coordinate planes are symmetry planes. Furthermore, it is invariant under a translation in $z$-direction of distance d. The planar ends are distance $d/2$ apart, and the planes asymptotic to the ends intersect the surface in pairs of orthogonal symmetry lines. When dividing the surface by its translational symmetry group, one obtains a minimal surface of genus 3 with two planar ends in $S^1 \times \mathbb{R}^2$.



Three fundamental pieces of $CHM_{1,1}$

We now give the Weierstraß data on the quotient surface.

Consider for some modulus $a > 0$ the Riemann surface $X$ given by

$$y^4 = x^2(x^2-1)(x^2-a^2)$$

Then the meromorphic 1-forms

$$\begin{aligned} Gdh &= \rho x^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}(x^2-a^2)^{-\frac{5}{4}}dx &&= \frac{\rho}{y}\frac{dx}{x^2-a^2}dx \\ \frac{1}{G}dh &= \frac{1}{\rho}x^{\frac{1}{2}}(x^2-1)^{-\frac{3}{4}}(x^2-a^2)^{\frac{1}{4}}dx &&= \frac{y}{\rho}\frac{dx}{x^2-1}dx \\ dh &= (x^2-1)^{-\frac{1}{2}}(x^2-a^2)^{-\frac{1}{2}}dx &&= \frac{x}{y^2}dx \end{aligned}$$

become single-valued on $X$ and we can write down the Weierstraß representation as

$$(*) \qquad z \mapsto \mathrm{Re}\int^z \left(\frac{1}{G} - G, i(G + \frac{1}{G}), 2\right)dh$$

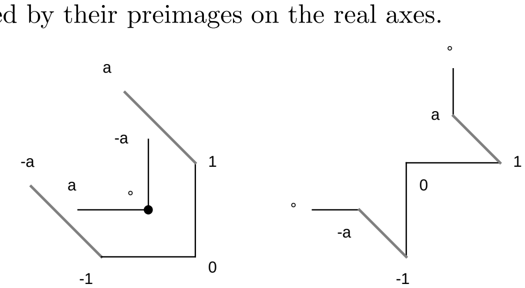
Note that in this setup, the Weierstraß data depend only on the modulus of the underlying Riemann surface (up to the Lopez-Ros parameter $\rho$, which is easily fixed for the current problem). This will remain true for the other surfaces we consider.

Observe that the symmetries of the surface are built into the defining equation and the 1-forms. In particular, the restriction of the Weierstraß parameterization to the upper half plane gives one quarter of a a fundamental piece of the surface, as cut out of the surface by its symmetry planes.

We now consider the flat metrics $ds = |Gdh|$ and $ds = |\frac{1}{G}dh|$, restricted to the upper half plane. With other words, we look at the two image domains of the upper half $x$-plane under the Schwarz-Christoffel maps

$$\begin{aligned} z &\mapsto \int^z Gdh \\ z &\mapsto \int^z \frac{1}{G}dh, \end{aligned}$$

as these flat metrics are just the pullback metrics of the euclidean metric under the Schwarz-Christoffel maps. Below are figures of these domains. They lie always to the left of the boundary curves, when these are followed anti-clockwise. The vertices are labeled by their preimages on the real axes.




The $Gdh$ and $\frac{1}{G}dh$ orthodisks for $CHM_{1,1}$

The fat dot within the left domain marks a branched point, the angle there is not $\frac{\pi}{2}$, but $5\frac{\pi}{2}$. The shaded diagonal segments correspond to the straight lines on the surface, while the vertical and horizontal edges correspond to intersections of the surface with the vertical symmetry planes. Note also the symmetry of the domains with respect to the $y = -x$ diagonal which corresponds to a rotational symmetry of the surface around a straight line lying on the surface.

We think of such a generalized planar polygon as the graphical representation of a (piece of a) Riemann surface together with a meromorphic 1-form on it.

If we glue copies of these polygons together by a translation at their (shaded) diagonal segments, we obtain a periodic infinite polygon which is an *orthodisk* in the sense of [WW2]. This periodic orthodisk represents now a periodic quarter piece of the surface (cut out by the vertical symmetry planes). The vertices of these orthodisks corresponds to points on the minimal surface where the surface is met by both vertical symmetry planes simultaneously. These are on one hand the ends, and on the other hand the saddle points with vertical normal.

In abuse of notation, we will also call the fundamental polygon above an orthodisk. Finally, the quotient of this periodic orthodisk by its translational symmetry will be called the corresponding *orthocylinder*.

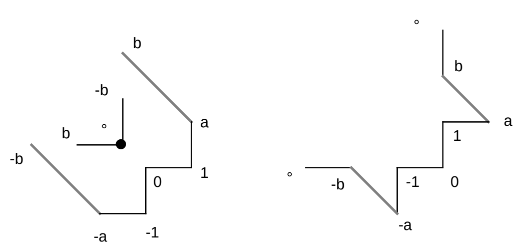
**Orthodisks for the general case.**

We will now modify the fundamental piece of the surface by adding further handles and parallel planar ends, while keeping (but not increasing) the dihedral symmetry. To do so, we will modify the $CHM_{1,1}$ orthodisk by introducing new vertices on the outer sheet boundary (which corresponds to handle addition for the surface) and new interior sheets (which corresponds to adding new planar ends).

Roughly speaking, the $Gdh$ orthodisk for $CHM_{m,n}$ has $2m + 1$ vertices in the outer sheet boundary and $n$ interior branched points, while the $\frac{1}{G}dh$ orthodisk for $CHM_{m,n}$ has $2m + 3$ vertices in the outer sheet boundary and $n - 1$ interior branched points.

Note that the angles of these orthodisks define the Weierstraß data completely, so giving the geometric shape of the orthodisks defines the candidate Weierstraß data.

Below are two examples, illustrating the addition of vertices in the outer sheet boundary:

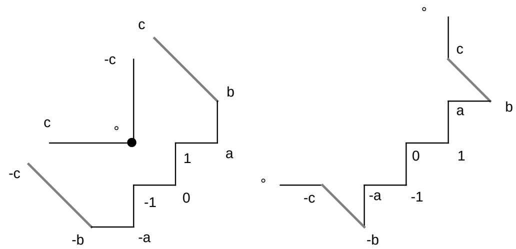



Orthodisks for $CHM_{1,2}$

corresponds to the Weierstraß data

$$Gdh = \rho x^{-\frac{1}{2}}(x^2 - 1)^{\frac{1}{2}}(x^2 - a^2)^{-\frac{3}{4}}(x^2 - b^2)^{-\frac{5}{4}}dx$$
$$\frac{1}{G}dh = \frac{1}{\rho}x^{\frac{1}{2}}(x^2 - 1)^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{4}}(x^2 - b^2)^{\frac{1}{4}}dx$$

and




Orthodisks for $CHM_{1,3}$

corresponds to the

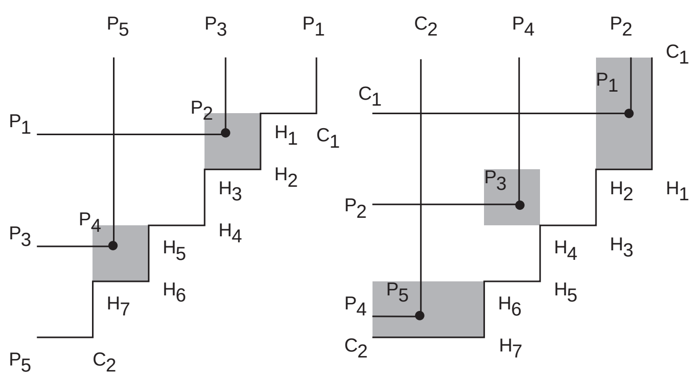
$$
\begin{aligned}
Gdh &= \rho x^{-\frac{1}{2}}(x^2-1)^{\frac{1}{2}}(x^2-a^2)^{-\frac{1}{2}}(x^2-b^2)^{-\frac{1}{4}}(x^2-c^2)^{-\frac{5}{4}}dx\\
\frac{1}{G}dh &= \frac{1}{\rho}x^{\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}(x^2-a^2)^{\frac{1}{2}}(x^2-b^2)^{-\frac{3}{4}}(x^2-c^2)^{\frac{1}{4}}dx
\end{aligned}
$$

In [WW2], we have defined the orthodisks for the $DH_{m,n}$ surfaces, and we will use this definition to create the $CHM_{m,n}$ orthodisks. To achieve this, we will describe a geometric procedure to pass from any pair of $DH_{m,n}$ orthodisks to a pair of $CHM_{m,n}$ orthodisks:

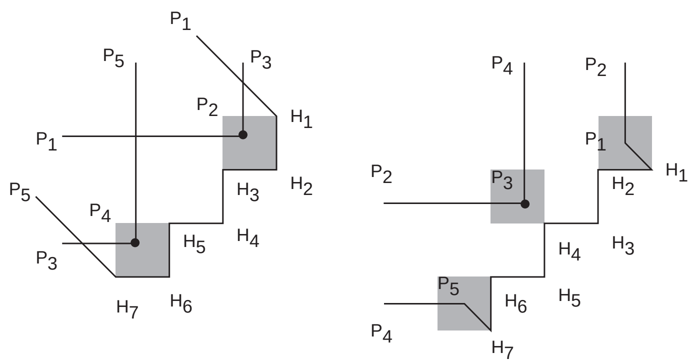
Take the $Gdh$ orthodisk and replace the segment $H_1C_1P_1$ by an infinite diagonal $H_1P_1$ to the upper left. Do the same symmetrically by replacing the segment $H_{2n+1}C_2P_{2m+1}$ by a diagonal $H_{2n+1}P_{2m+1}$. The orthodisk is identified along the two diagonals to give the orthocylinder.

Similarly, in the $\frac{1}{G}dh$ orthodisk, replace $H_1C_1P_1$ by a finite diagonal $H_1P_1'$ ending at a point $p'$ where this diagonal cuts the line through $P_1P_2$ and then going up to $P_2'$.

We illustrate this by the following figures (ignore the shaded areas for the moment):




A pair of orthodisks for $DH_{2,3}$




A pair of orthodisks for $CHM_{2,3}$

For the general $CHM_{m,n}$-surfaces we define using parameter points $1 = x_1 < x_2 < \dots x_{m+n}$

$$G(x) = x^{-\frac{1}{2}} \left( \prod_{i=1}^{n-1} (x^2 - x_i^2)^{\frac{(-1)^{i+1}}{2}} \right) (x^2 - x_n^2)^{\frac{(-1)^{n+1}}{4}} (x^2 - x_{n+1}^2)^{(-1)^n \frac{3}{4}}$$
$$\cdot \prod_{j=1}^{m-1} (x^2 - x_{j+n+1}^2)^{\frac{1}{2} - (-1)^j}$$
$$(**) \qquad dh = (x^2 - x_n^2)^{-1/2} (x^2 - x_{n+1}^2)^{-1/2}$$

*Remark.* Each $CHM_{m,n}$ surface is also a $CHM_{km,kn}$ surface for any positive integer $k$, by taking a $k$-fold copy of the $CHM_{m,n}$ fundamental piece. It is conceivable that there might possibly be other $CHM_{km,kn}$ surfaces than the natural one coming from $CHM_{m,n}$, but there is no indication whatsoever for this.

**Reformulation of the period problem.**

Now we explain how to use the orthodisk set-up to turn the *period problem* around.

Usually, one wants to determine the surface moduli so that the parameterization (*) becomes well-defined on the infinite singly-periodic Riemann surface, i.e. so that all periods of the 1-forms $(\frac{1}{G} - G)dh, i(G + \frac{1}{G})dh$ and $dh$ become *purely imaginary.*

This condition on the first two forms can be replaced formally by the requirement that the periods of $Gdh$ and $\frac{1}{G}dh$ become *complex conjugate.* This can be checked directly by looking at the orthodisks, because they were obtained by integrating just these 1-forms. Here it is important to note that the two orthodisks can be identified topologically by a homeomorphism mapping vertices to corresponding vertices.

**Definition 2.1.** The $Gdh$ and $\frac{1}{G}dh$ orthodisks are called *conjugate* if corresponding cycles develop to complex conjugate numbers. In our pictures, the 'real' axes fixed by that conjugation will always be the $y = -x$ axes. In particular, corresponding finite oriented edges have always to be conjugate.

Hence the period problem becomes the question whether a pair of conjugate orthodisks determines *the same surface.* This is the case if there exists a conformal map between the orthodisk domains mapping vertices to corresponding vertices. This map can then be continued by reflection to the whole surface.

For the periods of $dh$, it is also helpful to look at the image of the upper half $x$-plane under the map $p \mapsto \int^p dh$. The $dh$ forms are always very simple, see (**). By the Schwarz-Christoffel formula, the image domain is just a rectangle, giving rise to two periods. One is purely imaginary and not bothering us, and the other is purely real and giving the translational vector for the minimal surface. Thus the $dh$ differential causes no trouble for the $CHM_{m,n}$ surfaces.

Hence we have shown:

**Theorem 2.2.** *The period problem for $CHM_{m,n}$ can be solved if and only if there is a pair of conjugate $Gdh$ and $\frac{1}{G}dh$ orthodisks which are conformal in the sense that there is a conformal map between them mapping vertices to corresponding vertices.* □

The advantage of this reformulation of the period problem lies in the possibility to use Teichmüller theoretic methods to control the moduli by geometric deformations of the orthodisks.

## 3. Existence Proof: Geometric Coordinates

In this section we will introduce geometric coordinates for pairs of orthodisks and outline the existence proof for the $CHM_{m,n}$ surfaces, using the $DH_{m,n}$ surfaces as the model case.

The existence proof consists of several steps. The first is to set up a space of *geometric coordinates* such that each point in this space gives rise to a pair of conjugate orthodisks as described in section 2.

Given such a pair, one canonically obtains a pair of marked Riemann surfaces with meromorphic 1-forms having complex conjugate periods. If the surfaces were conformally equivalent, these two 1-forms will serve as the 1-forms $Gdh$ and $\frac{1}{G}dh$ in the Weierstraß representation.

After that, it remains to find a point in the geometric coordinate space so that the two surfaces are indeed conformal. To achieve this, a nonnegative *height function* $\mathcal{H}$ is constructed on the coordinate space with the following properties:

(1) $\mathcal{H}$ is proper;
(2) $\mathcal{H} = 0$ implies that the two surfaces are conformal;
(3) There is a $CHM_{1,1}$ surface;
(4) Given a $CHM_{m-1,n-1}$ or a $CHM_{m,n-1}$ surface, there is a smooth locus which lies properly in the $CHM_{m,n}$ coordinate space;
(5) On that locus, if $d\mathcal{H} = 0$, then actually $\mathcal{H} = 0$.

This setup separates local and global aspects of the proof: While the first three properties are global in nature, the last two are essentially local.

The height should be considered as some adapted measurement of the conformal distance of two surfaces. Hence it is natural to construct such a function using conformal invariants. We have chosen to built an expression using the extremal lengths of suitable cycles.

The first condition on the height poses a severe restriction on the choice of the geometric coordinate system: The extremal length of a cycle become zero or infinite only if the surface develops a node near that cycle. Hence we must at least ensure that when reaching the boundary of the geometric coordinate domain, *at least one* of the two surfaces degenerates *conformally*.

This condition is called *completeness* of the geometric coordinate domain.

Fortunately, we can use the definition of the geometric coordinates for $DH_{m,n}$ to derive complete geometric coordinates for $CHM_{m,n}$. In [WW2], there is a definition of a geometric coordinate domain of real dimension $m+n$ for the $DH_{m,n}$ surfaces. In the figure above, this domain is indicated by the shaded boxes wich restrict the positions of the interior branched points. Now observe that the procedure described in section 2 converts any pair of conjugate orthodisks in the geometric coordinate domain for $DH_{m,n}$ to a pair of conjugate orthodisks for $CHM_{m,n}$ and vice versa. The shaded boxes of $DH_{m,n}$ also restrict naturally to shaded boxes of $CHM_{m,n}$. Because we have dropped one parameter, the resulting coordinate domain has dimension $m+n-1$. The precise shape of the shaded boxes is actually completely determined by the completeness condition: The lower and right edges of a box are given by orthodisk edges, and the upper and left edges by corresponding right and lower edges of *conjugate* boxes.

Now using the arguments in [WW2] we obtain

**Theorem 3.1.** *This coordinate system for $CHM_{m,n}$ is complete.*

## 4. Existence Proof: The Height Function

The height function for $CHM_{m,n}$ is built using a slightly modified choice of the cycles for the height of $DH_{m,n}$. As we have reduced the dimension of the coordinate space by 1, we have to choose one cycle less, and we have to ensure that the chosen cycles are well defined.

The domains in which the cycles will be defined will be the orthocylinders which are obtained from the $CHM_{m,n}$ orthodisks by identifying the diagonal segments.

For a cycle $c$ connecting pairs of edges denote by $\operatorname{Ext}_{Gdh}(c)$ and $\operatorname{Ext}_{\frac{1}{G}dh}(c)$ the extremal lengths of the cycle in the $Gdh$ and $\frac{1}{G}dh$ orthocylinders. Recall that this makes sense as we have a natural topological identification of these domains (up to homotopy) mapping corresponding vertices onto each other.

The height function on the space of geometric coordinates will be a sum over several summands of the following type:

**Definition 4.1.** Let $c$ be a cycle. Define

$$\mathcal{H}(c) = \left| e^{1/\operatorname{Ext}_{Gdh}(c)} - e^{1/\operatorname{Ext}_{\frac{1}{G}dh}(c)} \right|^2 + \left| e^{\operatorname{Ext}_{Gdh}(c)} - e^{\operatorname{Ext}_{\frac{1}{G}dh}(c)} \right|^2$$

The rather complicated shape of this expression is required to prove the properness of the height function: Because there are sequences of points in the space of geometric coordinates which converge to the boundary so that *both* orthodisks degenerate for the same cycles, the above expression must be very sensitive to different rates with which this happens. Due to the Monodromy Theorem 4.1.2 in [WW2], it is sometimes possible to detect such rate differences in the growth of $\exp \frac{1}{\operatorname{Ext}(c)}$ for degenerating cycles with $\operatorname{Ext}(x) \to 0$.

The assumptions of the Monodromy Theorem impose certain restrictions on the choice of cycles for the height, and there are further restrictions coming from the Regeneration Theorem 4.5 below.

Now let's introduce the cycles formally:

We first introduce cycles related to the outer sheet:

$$\begin{aligned}
\alpha_2 &: H_{2n}H_{2n+1} \to P_{2m}H_{2n+1} = P_1H_1 \to H_2H_3 \\
\alpha_3 &: H_1H_2 \to H_3H_4 \\
\ldots &: \ldots \to \ldots \\
\alpha_{2n} &: H_{2n-2}H_{2n-1} \to H_{2n}H_{2n+1} \\
\alpha_{2n+1} &: H_{2n-1}H_{2n} \to H_{2n+1}P_{2m} = H_1P_1 \to H_1H_2
\end{aligned}$$

The first and the last cycle in this list are intended to cross the diagonal identification line and are best thought of as cycles in the corresponding orthocylinders where we also measure the extremal lengths

We group these cycles in pairs symmetric with respect to the $y = -x$ diagonal and also require that the cycles are symmetric themselves:

$$\gamma_i = \alpha_{i+1} + \alpha_{2n+2-i}, \qquad i = 1, \ldots, n-1$$

These cycles will detect degeneracies on the outer sheet boundary.

Next, we take as $\gamma$ the core cycle of the orthocylinder (which connects corresponding points on the identifying diagonals). This cycle will detect if one of the inner sheets degenerates towards the outer sheet boundary.

These cycles will be sufficient to build a proper height function. To guarantee that a vanishing height implies conformality of the pair of orthodisks, we add $m-1$ more cycles $\beta_j$ connecting each $P_jP_{j+1}$ and $P_{2m+2-j}P_{2m+1-j}$ to edges of the outer boundary for $j = 1, \ldots, m-1$. The edges on the outer boundary are chosen so that the extremal lengths of all cycles give conformal coordinates for the space of orthodisks (see lemma 4.3). We also want the cycles to satisfy the following technical conditions:

(1) they have two symmetric components which do not cross the $y = -x$ diagonal;
(2) they do not foot on one of the outer sheet boundary edges which emanates from the diagonal
(3) they do not foot on $H_1H_2$ or $H_{2n}H_{2n+1}$

The last two conditions are needed for the purpose of regeneration, as will be explained later on.

Using admissible cycles, the height will be composed as a sum of the following terms:

**Definition 4.2.** The height for the $DH_{m,n}$ surface is defined as

$$\mathcal{H} = \sum_{i=1}^{n-1} \mathcal{H}(\gamma_i) + \mathcal{H}(\gamma) + \sum_{j=1}^{m-1} \mathcal{H}(\beta_j)$$

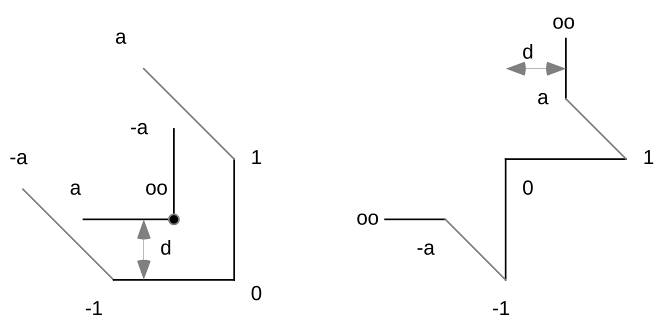
**Lemma 4.3.** *If $\mathcal{H} = 0$, the two orthodisks are conformal.*

*Proof.* Map the $Gdh$ orthodisk conformally to the upper half plane so that $H_{m+1}$ is mapped to 0, $P_{m+1}$ to $\infty$, and $H_m$ to 1. The vertices are mapped to points $x_k \in \mathbb{R}$ and the cycles are carried to cycles in the upper half plane which are symmetric with respect to the imaginary axes and which connect segments $x_k x_{k+1}$ to other segments $x_{k'} x_{k'+1}$. Note that there are only two cases of cycles: One type connects a segment on the positive axes to the symmetric segment on the negative axes, and the other consists of a pair of cycles, the first connecting two segments on the positive real axes and the second connecting the corresponding segments on the negative axes. Furthermore, all these cycles consist of symmetric curve families. Hence their extremal length can be computed as half of the extremal length of the portion of the cycle in the first quadrant. Using $z \mapsto z^2$ to map the first quadrant to the upper half plane, we see that each of the mapped half-cycles now connects just one pair of segments on the real axes, and hence determines the cross ratio of the endpoints of these segments. By the choice of the cycles, all possible cross ratios are determined. Hence the whole configuration of the $x_i$ is determined by the extremal lengths. The same holds for the $\frac{1}{G}$ orthodisk, and the claim follows. □

To illustrate the use of extremal lengths, we give a quick existence proof for $CHM_{1,1}$:

**Example.** Existence of $CHM_{1,1}$

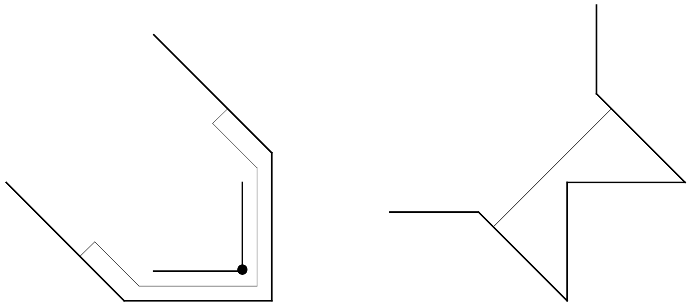
We begin by specifying the geometric coordinates. Below is a pair of conjugate orthodisks. We normalize the domains by fixing the length of the edge $(0,1)$ in the figure to 1 and then take the distance $d \in (0,1)$ of the edge $(-1,0)$ to the edge $(a,\infty)$ as the geometric coordinate. Recall that we have also imposed a reflectional symmetry along the $y = -x$ axes.




A geometric coordinate interval for $CHM_{1,1}$

To prove the existence of $CHM_{1,1}$, we need to find a pair of conjugate orthodisks which are also conformal.

From the definition of the height function, the only cycle which remains in this case is $\gamma$, connecting corresponding points on the edges $(1,a)$ and $(-1,-a)$:



The cycle $\gamma$ from the height function

Here, a direct extremal length argument using $\gamma$ is sufficient: If $d$ is small, $\mathrm{Ext}_\gamma(Gdh)$ is large while $\mathrm{Ext}_\gamma(\frac{1}{G}dh)$ remains bounded. The opposite holds for $d$ large. Hence there is a value of $d$ such that

$$\mathrm{Ext}_\gamma(Gdh) = \mathrm{Ext}_\gamma(\frac{1}{G}dh)$$

which was to be proven. $\square$

Martín and Rodríguez ([MaRo]) have proven that this period problem has indeed a unique solution, using direct methods.

In the general case, we have

**Theorem 4.4.** *The height function is proper on the space of geometric coordinates.* $\square$

The changes to the proof of the corresponding theorem 4.6.1 in [WW2] concern only the notation and are left to the reader. Note that the argument there relies heavily on the Monodromy Theorem 4.4.2, which remains valid in our setting, because the involved cycles connect edges of the same type as in [WW2].

The last part of the proof of the Main Theorem requires to prove the

**Regeneration Lemma 4.5.** *There is for a given $CHM_{m-1,n-1}$ or $CHM_{m,n-1}$ surface a certain good locus in the space of geometric coordinates for $CHM_{m,n}$ with the following properties:*

(1) *it lies proper within the space of geometric coordinates;*
(2) *if $d\mathcal{H} = 0$ on the locus, then actually $\mathcal{H} = 0$.*

This locus is defined by the requirement that all but one (if regenerating from $CHM_{m,n-1}$) or all but two (if regenerating from $CHM_{m-1,n-1}$) extremal lengths of the $Gdh$ and $\frac{1}{G}dh$ orthodisks are equal.

To show that this locus is nonempty requires a regeneration argument:

One constructs from an existing lower genus $CHM_{m-1,n-1}$ or a $CHM_{m,n-1}$ surface a point on the boundary of the geometric coordinate space of the $CHM_{m,n}$ surfaces. This is done purely geometrically.

Using an implicit function theorem, one then regenerates the surface into the geometric coordinate space to obtain a piece of the good locus. Another application of the implicit function theorem to interior points shows that the stratum lies in fact proper within the coordinate space.

In the last step, one computes the derivative of the height function with respect to certain natural deformations obtained from pushing edges of the orthodisks. Here it is important that on the stratum, the height function becomes much more simple and its derivatives can be handled.

Intuitively, pushing one edge inside the $Gdh$ domain (and decreasing its size) will result in pushing a corresponding edge of the $\frac{1}{G}dh$ domain outside, thus increasing its size. These changes should have monotone effects in opposite directions to suitable extremal lengths, and this can be proven by explicit computation of the derivatives of the extremal lengths under deformations of the domains.

It is important to note that all these arguments and computations take place locally and hence carry over to this case without *any* change.

This completes the outline of the proof of the Main Theorem.

## 5 Period Computation and Parametrization

In this section, we will explain techniques which were developed to solve the period problems for the $CHM_{m,n}$ surfaces numerically and to create suitable parameterizations for drawing the surfaces. These techniques are rather general and are also of theoretical interest.

The computation of the periods causes several difficulties: The integrands of the Weierstraß data are multivalued, and the periods are integrals over cycles in the complex plane. So it is tempting to replace these integrals by a real line integral from the punctures $x_i$ to $x_{i+1}$ which is correct if the integral converges. Unfortunately, some $x_i$ are non-integrable singularities.

One could try to implement analytic continuation of the integrands and path integration. However, in our cases it turns out that the points $x_i$ are often numbers which are rather close together so that integrating along a curve passing through a pair of such zeroes becomes hazardous.

The idea then is to replace the 1-forms above by cohomologous ones with integrable singularities and then to do the numeric integration using standard integrators. For this, one has to find the replacements *efficiently*.

Suppose that a Riemann surface $X$ is given by

$$y^n = P(x) = \prod_{i=1}^{2g+1} (x - x_i)^{d_i}$$

where the $x_i$ are real numbers and the $d_i$ are integers. This amounts to saying that the surface is a cyclically branched cover over the Riemann sphere punctured on the real axes. For instance, the $CHM_{1,1}$ surface is described by

$$y^4 = P(x) = x^2(x^2 - 1)(x^2 - a^2).$$

Suppose a meromorphic 1-form $\omega$ on $X$ is given by

$$\omega = T(x)\frac{dx}{y}$$

with some rational function $T(x)$ such that

$$F(x) = T(x)P(x)$$

is a polynomial of some degree $d$. We have in mind for $\omega = Gdh$ the rational function

$$T(x) = \frac{1}{x^2 - a^2}.$$

By the dimension of the meromorphic de Rham Cohomology group, there is a polynomial $A$ and a polynomial $R$ such that

$$\frac{F(x)}{y^{n+1}}dx - \frac{A(x)}{y}dx = d\frac{R(x)}{y} \tag{1}$$

So instead of $\int_\gamma \frac{F(x)}{y^{n+1}}dx$ we can now compute $\int_\gamma \frac{A(x)}{y}dx$, which can be done on a real line segment $x_i x_{i+1}$. The task is to find $A(x)$ explicitly.

By elementary computation, (1) is equivalent to

$$A = \frac{nF + P'R}{nP} - R' \tag{2}$$

so that we have to find $R$ with $\deg R \leq 2g$ such that

$$nF + P'R \equiv 0 \pmod{P} \tag{3}$$

Because our $F$ will always share many zeroes with $P$, one can easily to compute $R$ by Lagrange interpolation using the fact that

$$R(x_i) = -n\frac{F(x_i)}{P'(x_i)}$$

We give the results of the computation in the simplest case:

**Example.** The $CHM_{1,1}$ surface.

Here we have for $Gdh$

$$\begin{aligned} P(x) &= x^2(x^2-1)(x^2-a^2) \\ F(x) &= x^2(x^2-1) \\ R(x) &= -2\frac{x(x^2-1)}{a^2(a^2-1)} \\ A(x) &= \frac{3x^2-a^2}{a^2(a^2-1)} \end{aligned}$$

The form $\frac{1}{G}dh$ does not cause problems.

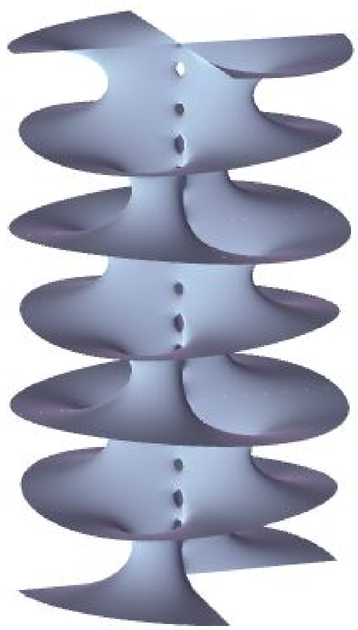
For increasing genus, the rational functions become algebraically rather complicated.

The second interesting problem related to the surfaces here is to find a parameterization which gives acceptable pictures. Using the eightfold symmetry of the fundamental domain of the surfaces, it is only necessary to evaluate the Weierstraß parameterization in (say) the first quadrant of the euclidean plane.

One minor obstacle arising here is that one has to compute complex line integrals in this domain over a square root of a polynomial of high degree. This leads rather quickly to problems with the branched cuts of any simple square root implementation. In our case one can just factor the polynomial and take the square root of each factor, where it is well defined.

The positive roots of the polynomial $P(x)$ represent special points on the surface which must be paid attention to: Some of them are saddle points with vertical normals, which should be met by a coordinate line, and the others represent the ends. The latter cause the most serious problems, because an appropriate parameterization of an end should use polar coordinates centered at the puncture. So the idea is to use coordinates on the first quadrant which look like polar coordinates around the singularities corresponding to ends.

This approach becomes more and more difficult the more ends are represented by points on the real axes. For the $CHM_{1,n}$ surfaces, this is just one point, and a simple parametrization leads to



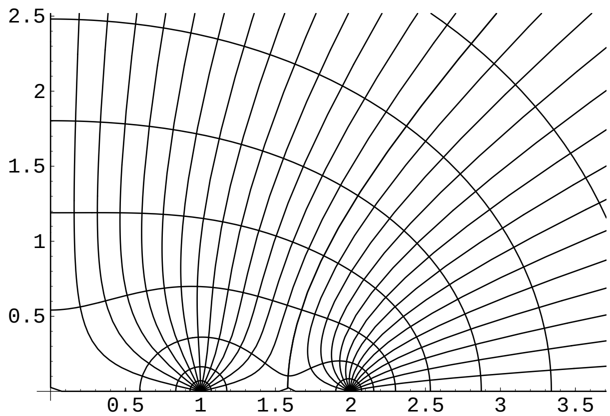
Three fundamental pieces of $CHM_{1,5}$

For $CHM_{2,3}$ there are two ends represented by points on the positive real axes, and the construction of suitable coordinates becomes more complicated:

Consider

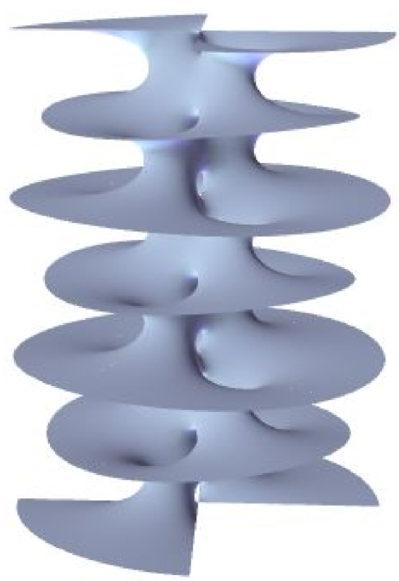
$$f : z \mapsto \int^z \frac{1}{w-b} + \frac{1}{w-c} dw$$

The lines $\mathrm{Re} f(z) = const$ and $\mathrm{Im} f(z) = const$ are the parameter lines which we want on the first quadrant. So we need to invert $f$ and plot the two branches separately. This is explicitly possible and yields the following coordinate lines. Note that we have chosen $b = 1$ and $c = 2$ for better illustration.




Coordinate lines for $CHM_{2,3}$

One of the motivating questions behind this paper was how the handles might be distributed in height between the planar ends. It was somewhat a surprise that adding a single handle to $CHM_{1,1}$ produces a singly periodic surface whose planar ends have saddles which have alternatingly genus 1 and genus 2:



Three fundamental pieces of $CHM_{1,2}$

Another question was to gain some insight how the additional handles in the $CHM_{1,n}$ surface might be arranged. From the Weierstraß data it was clear that handles have to be aligned atop of each other, but no information was available about their relative sizes. It was even conceivable that there would be thinner and thinner handles for growing value of $n$. Even though the $CHM_{1,5}$ surface is certainly no evidence for the general picture, there is at least a slight indication that the handle towers between two parallel ends resemble a portion of the singly periodic Scherk tower. At the moment, there is no proof of this.

It is also worth noting that the horizontal elongation of the cylindrical pieces of $CHM_{1,1}$ between consecutive ends become much more pronounced for higher genus.

## References


[**CHK**] M. Callahan, D. Hoffman, H. Karcher, *A Family of Singly Periodic Minimal Surfaces Invariant under a Screw Motion*, Experimental Mathematics **2** (1993), 157–182.

[**CHM**] M. Callahan, D. Hoffman, W.H. Meeks III, *Embedded Minimal Surfaces with an infinite number of ends*, Inventiones Math. **96** (19989), 459–505.

[**HK**] D. Hoffman, H. Karcher, *Complete Embedded Minimal Surfaces of Finite Total Curvature*, Geometry V (R. Osserman, ed.), Encyclopaedia of Mathematical Sciences, vol. 90, Springer, Berlin Heidelberg New York Barcelona Budapest Hong Kong London Milan Paris Santa Clara Singapore Tokyo, 1997.

[**HKW1**] D. Hoffman, H. Karcher, F. Wei, *The genus one helicoid and the minimal surfaces that led to its discovery*, Global Analysis and Modern Mathematics (K. Uhlenbeck, ed.), Publish or Perish Press, 1993, pp. 119–170.

[**HKW2**] D. Hoffman, H. Karcher, F. Wei, *The Singly Periodic Genus-One Helicoid*.

[**Ka1**] H. Karcher, *Embedded Minimal Surfaces derived from Scherk's example*, Manuscripta Mathematica **62** (1982), 83–114.

[**Ka2**] H. Karcher, *Construction of minimal surfaces*, Surveys in Geometry, University of Tokyo, 1989, pp. 1–96.

[**MaRo**] Francisco Martín, Domingo Rodríguez, *A Characterization of the Periodic Callahan-Hoffman-Meeks Surfaces in Terms of Their Symmetries*, Duke Math. Journal **89** (1997), 445–463.

[**MR**] W. Meeks III, H. Rosenberg, *The Geometry of Periodic Minimal Surfaces*, Comm. Math. Helv. **68** (1993), 538–578.

[**Meu**] J. B. Meusnier, *Mémoire sur la courbure des surfaces*, Mémoires des savantes étranges **10** (1785), 477-510.

[**R**] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, Abh. Königl. Ges. d. Wiss. **13** (1867), Göttingen, 3–52.

[**Sch**] H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen*, J. R. Angew. Math. **13** (1835), 185–208.

[**W**] M. Weber, *On the Horgan Minimal Non-Surface*, Calc. Var. **7** (1998), 373–379.

[**WW1**] M. Weber and M. Wolf, *Minimal Surfaces of Least Total Curvature and Moduli Spaces of Plane Polygonal Arcs*, GAFA **8** (1998), 1129–1170.

[**WW2**] M. Weber and M. Wolf, *Teichmüller Theory and Handle Addition for Minimal Surfaces*, Preprint no 555, SFB 256, Bonn.



Mathematisches Institut Bonn, Beringstrasse 1, 53115 Bonn, Germany
*E-mail address*: weber@math.uni-bonn.de